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Abstract of thesis entitled

"LMI Conditions for Robust Consensus of Uncertain Nonlinear Multi-agent Systems"

Submitted by

Dongkun Han

for the degree of Doctor of Philosophy

at The University of Hong Kong

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Establishing consensus is a key problem in multi-agent systems (MASs). This thesis proposes a novel methodology based on convex optimization in the form of linear matrix inequalities (LMIs) for establishing consensus in linear and nonlinear MAS in the presence of model uncertainties, i.e., robust consensus.

Firstly, this thesis investigates robust consensus for uncertain MAS with linear dynamics. Specifically, it is supposed that the system is described by a weighted adjacency matrix whose entries are generic polynomial functions of an uncertain vector constrained in a set described by generic polynomial inequalities. For continuous-time dynamics, necessary and sufficient conditions are proposed to ensure the robust first-order consensus and the robust second-order consensus, in both cases of positive and non-positive weighted adjacency matrices. For discrete-time dynamics, necessary and sufficient conditions are provided for robust consensus based on the existence of a Lyapunov function polynomially dependent on the uncertainty. In particular, an upper bound on the degree required for achieving necessity is provided. Furthermore, a necessary and sufficient condition is provided for robust consensus with single integrator and nonnegative weighted adjacency matrices based

on the zeros of a polynomial. Lastly, it is shown how these conditions can be investigated through convex optimization by exploiting LMIs.

Secondly, local and global consensus are considered in MAS with intrinsic nonlinear dynamics with respect to bounded solutions, like equilibrium points, periodic orbits, and chaotic orbits. For local consensus, a method is proposed based on the transformation of the original system into an uncertain polytopic system and on the use of homogeneous polynomial Lyapunov functions (HPLFs). For global consensus, another method is proposed based on the search for a suitable polynomial Lyapunov function (PLF). In addition, robust local consensus in MAS is considered with time-varying parametric uncertainties constrained in a polytope. Also, by using HPLFs, a new criteria is proposed where the original system is suitably approximated by an uncertain polytopic system. Tractable conditions are hence provided in terms of LMIs. Then, the polytopic consensus margin problem is proposed and investigated via generalized eigenvalue problems (GEVPs).

Lastly, this thesis investigates robust consensus problem of polynomial nonlinear system affected by time-varying uncertainties on topology, i.e., structured uncertain parameters constrained in a bounded-rate polytope. Via partial contraction analysis, novel conditions, both for robust exponential consensus and for robust asymptotical consensus, are proposed by using parameter-dependent contraction matrices. In addition, for polynomial nonlinear system, this paper introduces a new class of contraction matrix, i.e., homogeneous parameter-dependent polynomial contraction matrix (HPD-PCM), by which tractable conditions of LMIs are provided via affine space parametrizations. Furthermore, the variant rate margin for robust asymptotical consensus is proposed and investigated via handling generalized eigenvalue problems (GEVPs).

For each section, a set of representative numerical examples are presented to demonstrate the effectiveness of the proposed results.

An abstract of exactly 473 words

LMI Conditions for Robust Consensus of Uncertain Nonlinear Multi-agent Systems

by

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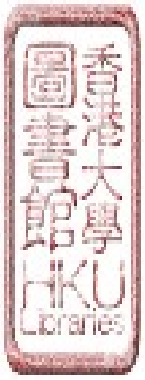

Declaration

I declare that the thesis and the research work thereof represent my own work, except where due acknowledgement is made, and that it has not been previously included in a thesis, dissertation or report submitted to this University or to any other institution for a degree, diploma or other qualifications.

Signed .....

Dongkun Han

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List of Publications

Journal papers:

1. D. Han, G. Chesi. H-infinity Analysis and Synthesis for Robust Synchronization: A Sum-of-squares Approach, In preparation.

2. D. Han, G. Chesi. Robust Synchronization via Homogeneous Parameter-dependent Polynomial Contraction Matrix. IEEE Transactions on Circuits and Systems I, Accepted.

3. D. Han, G. Chesi, C. K. Luk. Homogeneous Polynomial Lyapunov Functions for Robust Local Synchronization with Time-varying Uncertainties. IET Control Theory and Applications, 8(10), 855-862, 2014

4. D. Han, G. Chesi. Synchronization Conditions for Multi-agent Systems with Intrinsic Nonlinear Dynamics. IEEE Transactions on Circuits and Systems II, 60(4): 227-231, 2013.

5. D. Han, G. Chesi. Robust Consensus for Uncertain Multi-agent Systems with Discrete-time Dynamics. International Journal of Robust and Nonlinear Control, first published online: 24 Jan 2013.

6. D. Han, G. Chesi, Y. S. Hung. Robust Consensus for a Class of Uncertain Multi-agent Dynamical Systems. IEEE Transactions on Industrial Informatics, 9(1): 306-312, 2013.

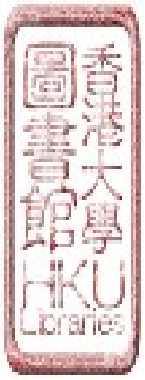
Conference papers:

1. D. Han, G. Chesi. Robust Discrete-time Consensus of Multi-agent Systems with Uncertain Interaction. 2012 IEEE International Conference on Control Applications, pages 1136-1141, Croatia, 2012.

2. D. Han, G. Chesi, Yeung Sam Hung. Synchronization Seeking in Multi-agent Dynamic Systems with Communication Uncertainties. 2011 IEEE International Symposium on Computer-Aided Control System Design, pages 656-661, Denver, CO, 2011.

List of Abbreviations

MAS	Multi-agent System
UAV	Unmanned Air Vehicles
LMI	Linear Matrix Inequality
SMR	Square Matrix Representation
MSF	Master Stability Function
SOS	Sum of Squares
QLF	Quadratic Lyapunov Function
HPLF	Homogeneous Polynomial Lyapunov Function
PLF	Polynomial Lyapunov Function
GEVP	Generalized Eigenvalue Problem
HPD-PCM	Homogeneous Parameter-dependent Polynomial Contraction Matrix



List of Symbols

$\mathbb{N}, \mathbb{R}, \mathbb{C}$	spaces of natural, real, and complex numbers
$\mathbb{R}^n$	n-dimensional Euclidean space
$\mathbb{R}^{n\times n}$	set of $n \times n$ real matices
A^T	transpose of matrix A
$\mathrm{he}(A)$	$A + A^T$, with $A \in \mathbb{R}^{n\times n}$
$A > 0$ $(A \geq 0)$	symmetric positive definite (semidefinite) matrix A
0_n	origin of $\mathbb{R}^n$
1_n	a column vector with all entries being 1, i.e., $(1, \ldots, 1)^T$
I_n	$n \times n$ identity matrix
$\mathrm{img}(A)$	image of matrix A
$\ker(A)$	null space of matrix A
$\mathrm{adj}(A)$	adjoint of A
$\mathrm{trace}(A)$	trace of matrix A
$A \otimes B$	Kronecker product of matrices A and B
$\mathrm{diag}(A_1, \ldots, A_n)$	block diagonal matrix with $A_1, \ldots, A_n$ on the diagonal
$\mathrm{spc}(A)$	set of eigenvalues of A $\in \mathbb{R}^{n\times n}$, i.e., $\mathrm{spc}(A) = \{\lambda : \det(\lambda I_n - A) = 0\}$
$\mathrm{LCM}\{a, b, \ldots\}$	least common multiplier of $a, b, \ldots$.
$\mathrm{co}\{X_1, \ldots, X_p\}$	convex hull of matrices $X_1, \ldots, X_p \in \mathbb{R}^{m\times n}$
$\mathrm{sq}(\theta)$	$(\theta_1^2, ..., \theta_a^2)^T \in \mathbb{R}^a$, $\theta \in \mathbb{R}^a$
$(*)^T AB$	$B^T AB$
$\lvert\cdot\rvert_i$	vector norm i on Euclidean space ($\lvert\cdot\rvert$)
$\lVert A\rVert_i$	induced matrix norm i of matrix A, i.e., $\lVert A\rVert_i = \max_{\lvert x\rvert_i=1} \frac{\lvert Ax\rvert_i}{\lvert x\rvert_i}$
$\mu_i(A)$	one-sided directional derivative of $\lVert\cdot\rVert_i$ in the direction A, i.e., $\mu_i(A) := \lim_{\epsilon\to 0} \frac{(\lVert I + \epsilon A\rVert_i - 1)}{\epsilon}$
$X^{[i]}$	i-th Kronecker power, i.e., $X^{[i]} = \begin{cases} X \otimes X^{[i-1]} & \text{if } i > 1 \\ 1 & \text{if } i = 0 \end{cases}$

Chapter 1

Introduction

1.1 Background

More than 2000 years is the history of control systems applied in human society. One of the earliest proofs is the water clocks described by Vitruvius (270 B.C.) [1]. Past several decades have witnessed a fast development of this engineering discipline from classical control theory to now sophisticated modern control theory. Instead of transformation methods in frequency domain dealing with single-input-single-output systems, modern control theory uses the state-space methods in time domain for multiple-input-multiple-output systems. With increasingly efficient methods for matrix computation and growingly powerful microprocessor for data processing, modern control theory is able to handle more complicated problems human encountered, like large-scale systems and complex systems.

When the corresponding advances come to the technologies of information transmitting, sensing and processing, they speed up the development of autonomous systems in uncertain environments, and also give rise to new opportunities and challenges for studying the cooperative multi-agent systems (MASs). In fact, the definition of MASs is initially investigated by collective behaviors widely occurring in biology and life science, e.g., bird flocking, fish schooling and bud swarming. These behaviors gain some properties that single agent can not achieve and provide extra benefits for individuals such as finding food, increasing the speed of flying and

escaping from predators. Thus, numerous attentions have been obtained for studying this cooperative behavior. One of the famous pioneering work in this field is the work of Beynold in which a computer model, called "Boids", is used to animate the cooperative behavior of fish schools [2]. Also in this work, three basic rules are given for this model: cohesion for flock centering, separation for collision avoidance and alignment for velocity matching. These properties are deemed as basic characteristics of a biological band. After that, based on a discrete-time approximation system, Vicsek proposes another multi-agent model for a group of autonomous particles traveling at a certain speed. Then this result is explained and expanded by Jadbabaie et al. in [3]. From then on, a general understanding is established that coordination of MASs is a result of biological principles with environmental information and interactions among individuals in this group.

In the field of control systems and computer science, any individual that can autonomously behave in its environment is called an agent. Though extensive debate is over the meaning of this term, we would like to give following definition similar with one given in [4].

Definition 1.1 *(agent) An agent is a single system in some environment where it is capable of autonomous actions in this environment in order to meet its designed objectives.*

Owning to the broad nature of above definition, we would like to further elaborate on some of its essential issues. Firstly, agent is considered as a single system which may refer to artificial entities, like computer, robots and electronic devices, and may also bear on biological entities, like fishes, birds, insects. Moreover, there is no specific environment required for MASs, as it can refer to a broad range of settings, where autonomous agents behave and interact with each other. Influence of human or outside intervention is not considered in order to keep the independence. Finally, an objective is designed for the group of agents. No specific method is given by this definition that how the cooperative goal can be achieved. Certain objectives can be very simple and just need reactive agent, such as a group of flights gather

at a certain airport, while some objectives are complicated with sophisticated agent model, such as a group of flights keep in a certain formation by communication with each other. In other words, agent has to execute individual autonomous actions in a target-oriented manner and vulnerable to its environment. Different with [2], Jennings proposes three characteristics for each autonomous agent to achieve the designed objectives [4]:

- Reactivity: the capacity to comprehend their environment and timely react to changes of the environment.
- Proactivity: the capacity to take initiative independently and perform goal-oriented behavior.
- Social ability: the capacity to communicate and influence with other agents.

Based on the definition of *agent*, it is easy to understand another related definition *multi-agent systems*. A great number of biological and artificial systems are composed of a group of agents interacting for a common goal. Examples can be easily found in World Wide Web, formation control of UAV (Unmanned Air Vehicles), remote control of AUVs (autonomous underwater vehicles), networked control of robotic teams, to name just a few. Multi-agent systems (MASs) are common in nature and can be explained in a way widely adopted in engineerings as follows [4]:

Definition 1.2 *(Multi-agent System) A MAS is a group of networked agents that behave together to achieve common goals.*

Currently, various topic of MASs are hot focus for sorts of academic societies, such as cooperative learning, coordination, dependability and fault-tolerance, whereas the most interesting issue for us is the consensus problem of MASs.

Consensus problem has been extensively studied in control theory and computer science in past decades . Meanwhile, kinds of powerful tools, like algebraic graph theory and contraction theory, have been developed to investigate consensus problem of MASs, as a beneficial circle, generating growing application and attracting

increasing academic interests and resources in this field. Here we give the definition of consensus which is widely adopted in control systems. Consensus means that all the states of a MAS are able to reach a certain agreement progressively. The states of MASs for this agreement could be artificial variables like computer virtual state, logical state, or kinds of physical terms such as energy, temperature, altitude, position, velocity, angle, etc. The mathematical description of consensus phenomenon with various dynamics will be introduced in the following chapters. It is worth noting that the consensus problem can be converted to a stability problem of the error dynamics (or called disagreement system) which is constructed based on original dynamics (a typical example of this transformation is given by Lemma 3.4). With a wide applications, consensus has already been employed to cope with many practical problems, such as formation control, flocking, rendezvous problems, attitude alignment [5–11]. Fig. 1.1 gives one application in formation control.

1.2 Literature Review

In this subsection, past researches on consensus problem will be reviewed in different models of MASs. Firstly, consensus problem of MASs with linear dynamics will be reviewed in fixed topology and varying topology. Then, the past works in consensus problem with nonlinear dynamics of complex network will also be studied.

1.2.1 Consensus in Networks with Fixed Topology

For the pioneering works on this topic, consensus problem of MASs arises and is investigated in computer science, where academic priority is given to computational algorithm [12, 13]. After that, a typical consensus protocol is studied on the headings and directions of moving particles with same speed [14]. In [3], the average consensus problem for MASs is investigated in the undirected communication network where a sufficient condition is proposed that consensus can be achieved as

(a)

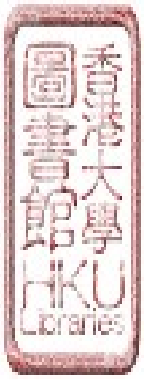
(b)

Fig. 1.1: One application of consensus: formation control.

long as each agent is jointly connected to all the others during the contiguous time intervals. In [9], a method of eigenvalue analysis of Laplacian matrix is proposed and the connectivity between topological connectivity and weighted adjacency matrix are established. It also proposes consensus condition of MASs with network communication constraints and information disturbs. Based on the work [3], Ren and Beard investigate the consensus problem in a more general case with directed information exchange. Also by combining graph theory and matrix theory, they provide a consensus condition that a spanning tree in communication network is a necessary and sufficient condition for consensus. Then, this method is extended to switching topology and asymptotical consensus condition is provided that consensus of MASs with switching topology can be achieved if it has a spanning tree frequently enough for communication network [15]. Thus, consensus of MASs with fixed topology has already been extensively investigated and non-conservative consensus conditions has been provided by checking the topology of network based on employing graph theory.

Linear dynamics of agent is widely adopted in that it can simply describe a real world phenomenon and give a solution for these mathematical problems based on linear algebra. Lots of existing results use this model and apply it in numerous real-world implementations like unmanned flying vehicles, kinetic gears and moving particles [3, 14]. Starting with initial works on linear dynamics, the consensus problems begin to attract attentions from a number of researchers. [9, 15]. Especially, the consensus problem of MASs with first-order dynamics is studied extensively where the agent is driven by the influence of its neighbours [16–18]. For first-order consensus, it is shown that the general communication structure plays a significant role for MASs to reach the asymptotical consensus.

Besides the first-order consensus, in order to meet the demand of practical implementations, a more complicated model of MASs with second-order dynamics is proposed and investigated [19–21]. In this model, not only position state is considered, but also the velocity state. The second order consensus requires that both



the position state and the velocity state of each agent tend to be same. This kind of dynamics gives a better approximation of motion in physics, thus triggering a great academic passion in this field. In [20], consensus condition for second-order dynamics is proposed by investigating the topological structure and the second largest eigenvalue of Laplacian matrix is deemed as a key role in second-order consensus. In [21], necessary and sufficient conditions for second-order consensus are proposed by eigenvalue analysis and matrix theory.

Furthermore, a more general model, including higher order consensus protocols, is proposed and considered in [22–24]. In [25], a general protocol for higher order consensus is considered according to the transverse stability to the manifold of consensus. This work is originated from the work of synchronization in complex networks [26]. In [22], a distributed containment problem is considered for networked Lagrangian systems with multiple leaders under a directed graph. In [24], the definition of subsystem is introduced and a necessary and sufficient condition is provided for asymptotical consensus with general higher-order dynamics if all subsystems are proven to be asymptotically stable. Moreover, this work also shows that for higher-order consensus, the largest number of disconnected stable and unstable consensus regions are provided and consensus of MASs can be achieved if the nonzero eigenvalues of the Laplacian matrix locate in the stable consensus regions.

Not merely consensus problems with continuous-time dynamics are investigated, but also the problems with discrete-time dynamics, where difference equation is used to describe the dynamics of MASs [27, 28]. In [27], discrete-time dynamics is considered for MASs with time-varying delays and a class of effective consensus protocols are provided to solve consensus problems with the assumption that the agent can merely use delayed information of themselves. In [28], both the cases of leaderless consensus and leader-follower consensus are considered for linear discrete-time MASs. By using state feedback protocols, these two consensus problems can be tackled, provided that the dynamic graph remains jointly connected. In [29], a distributed algorithm for average-consensus with discrete-time dynamics

is proposed based on a formal matrix limit notation of average-consensus, which can be achieved if at every instant the network topology is balanced and the union of graphs over every time interval is connected.

1.2.2 Consensus in Networks with Changing Topology

For consensus problem of MASs with fixed topology, non-conservative results can be given based on graph theory and matrix theory. But for the case of network with varying topology, the situation is the same. As we all know, perturbations and disturbances are brimming over this world. For a straightforward instance in electrical power grid, the parameters of power transmission lines, such as the values of resistance and capacitance, are fairly vulnerable to alter under inconsistent temperature and air pressure. Thus, numerous attentions have recently been casted on robust consensus problem of MASs with unfixed communication topology.

Firstly, the interaction topology between agents is assumed to be dynamically changing and can be described by a set of directed graphs [3, 14, 15, 30]. Vicseck et al. propose a discrete-time model for MASs with all agents moving in the plane. In this model, each agent's heading is driven by a local informations from each agent's information and its own [14]. This nearest neighbour rule is studied and extended to a more general case in which possible changes in nearest neighbours are taken into account over time [3]. A collection of simple graphs on n vertices is used to describe all possible neighbour relationships. Thus, a switched linear system is established to present the Vicseck model. Consensus condition is provided that a common steady state of each agent can be obtained if all agents are linked with their neighbours with sufficiently large frequency. In [30], same model of switching topology is studied and a common Lyapunov function of disagreement dynamics is established for this hybrid system. Based on same switching strategy, in [27], time-varying delays and switching communication topology are considered, where network topology switches amongst a group of graphs already known. A class of effective consensus protocols are provided by using the same state information at

double time-steps. In [28], via state feedback control protocols, consensus problems of both the leaderless and leader-following cases are investigated.

Secondly, networks with time-varying topology can be found and used in a great number of academic fields, like engineering, biological and social systems. This kind of network topology has successfully described the on-off state of communication links, the loss in data transmitting, the variations of topological parameters and the reconfiguration of formations in flocking problem. Thus, time-varying topology of MASs is investigated in recent years [31–34]. Consensus conditions are provided for MASs with time-varying topology without any time-delay or stochastic disturbances [32,33]. In [31], a time-varying topology with distributed stochastic approximation is used for describing the communication noises. Sufficient conditions are provided for mean square average consensus and sufficient condition is also given for almost sure consensus. In [34], asynchronous consensus problem of MASs with second-order dynamics is considered. Each agent can gain position and velocity information from others at each sampling-time. Sufficient condition is proposed for consensus of MASs with intermittent information transmission and asynchronous data update. In [35], the consensus problem of MASs with second-order dynamics in discrete-time is considered where the interaction topology is time-varying. By applying Lyapunov direct method, a consensus controller is designed for any bounded time-delays.

Furthermore, another effective way to model the time-varying networks is by using stochastic switching networks [36–40]. In [41], a blinking model of small-world network is proposed for nonlinear consensus and for this model, a connection graph stability method is used for synchronization problems. By using the same method, in [36], a new changing topology is considered where both fixed $2K$-nearest neighbor coupling and time-dependent on-off coupling are studied in this work. In [37], each agent is deemed as a random walker and location changes of agents in the lattices is used to model the random changes of network where agent can obtain information only from other agents in the same lattice. In [38], by employing

stochastic Lyapunov stability theory, sufficient conditions are provided for global synchronization with a high frequency and random switching network. In [39, 40], another stochastically blinking topology is considered in which coupling parameters of network randomly switch within a collection of values at a certain interval.

Lastly, topological uncertainties are also used to model the time-varying topology of MASs. By introducing this kind of uncertainty, networks with parametric uncertainty on topology can be effectively approximated in many real-life applications [42]. Especially for consensus problem, successful implementations are existing in a wide range of academic societies and industries [43–45]. In [43], consensus problem with linear dynamics is considered with additive uncertainty. In [44], robust synchronization problem is investigated where the network is disturbed by relative-attitude error. In [45], 2-D synchronization problem is analyzed where the control gains are under the disturbs of square integrable bounded and time-varying uncertainty. In [46], robust synchronization problem is considered by using contraction theory. The time-invariant parametric uncertainty is used to model the structural disturbances. For this kind of uncertainty, a constant contraction matrix is introduced to solve the robust synchronization problem.

1.2.3 Consensus in Networks with Nonlinear Dynamics

In various phenomenons, autonomous agents are ususally governed by intrinsic nonlinear dynamics where a comparatively rich complex behaviour can be exhibited, like periodic circles, bifurcation and chaotic manifold. Another definition: synchronization is defined to investigate these phenomenons. Synchronization problem is a key issue in engineering, biology, physics and social science. It usually considers the agent with nonlinear dynamics in complex networks. Actually, the synchronization and consensus with nonlinear dynamics are the same thing but with different names in corresponding societies [26, 47]. Joint examples can be easily found in World Wide Web, social science, biological Metabolism and power grids [26, 47–51]. A great number of methods have been invited and developed

both for nonlinear consensus of MASs and for chaotic synchronization [11,52]. We will use consensus with nonlinear dynamics instead of synchronization in the rest of this thesis.

Firstly, local consensus with nonlinear dynamics is extensively investigated in past decade from the pioneering work of Pecora and Carroll [53]. Specifically, for uncertain coupling matrix, master stability function method (MSF) is applied for local consensus with nonlinear dynamics in which a maximum Lyapunov exponent of differential equation for nonlinear network is calculated [49]. This method has been proven to solve the local consensus with nonlinear dynamics successfully by linearizing the nonlinear intrinsic function. Particular interest in local synchronization is triggered by this method and lots of works extend this method to various specific implementations such as kinds of clustering coefficients, sorts of coupling strength and different consensus protocols [54]. Besides the methods of eigenvalue analysis on the coupling matrix, another interesting and stimulating thought, named as Connection Graph Stability, is proposed by combing Lyapunov directed method with graph theory and applying this method to stochastic switching network [55]. Local consensus problems are also considered in time-varying topology. In [56], a consensus criteria is given for fast switching network with a sufficiently large switching rate. [57], by using a time-average topology to approximate the time-varying topology, consensus conditions are provided that the time-varying topology and the time-average topology are actually the same for commuting Laplacians.

In addition, a growing number of researches have been studying the global consensus problem of MASs with intrinsic nonlinear dynamics [25,47,58–63]. In [47], global consensus in complex networks is considered. By defining a generalized algebraic connectivity, the global convergence properties of obtaining consensus are analyzed in strongly connected networks. In [58], many conclusions have been obtained under the assumption that the network is weighted balanced, and a distributed algorithms is proposed by non-smooth analysis. In [60], A necessary and sufficient condition on convergence of a multiplicative sequence of reducible row-

stochastic matrices is proposed and an overall closed loop system is constructed to exhibit cooperative behaviors. For MAS with intrinsic nonlinear dynamics, Lyapunov methods are successfully applied to derive consensus conditions [47, 59, 62]. Particularly, quadratic Lyapunov function is a main approach widely employed to guarantee the consensability of MASs.

For a more sophisticated case of global consensus with nonlinear dynamics and time-varying network, Lyapunov stability method, especially the quadratic stability, is used to generate consensus conditions [43–45, 64]. According to uncertain adjacency matrix, MSF and eigenvalue analysis, or relevant derivative tools, can hardly be used, making Lyapunov stability theory as a mainstream method for robust consensus with nonlinear dynamics. In [65], impulsive consensus criteria is proposed for uncertain dynamical network with nonlinear intrinsic dynamics where the network coupling functions is assumed to be unknown yet bounded, provided that Lipschitz-like conditions are satisfied both for the intrinsic nonlinearity and for the coupling nonlinearity.

Consensus problem with nonlinear dynamics is also widely considered under the influence of topological uncertainties [43–46, 64]. In [43], robust consensus of MASs is investigated with additive uncertainty and consensus condition in terms of Linear Matrix Inequalities (LMIs) is provided by using matrix theory. Furthermore, a feedback controller is designed for robust consensus under bounded perturbations based on Riccati equations. In [44], provided that system dynamics is under the disturb of a relative-attitude error, a robust global consensus problem is analyzed with a decentralized hybrid feedback control scheme. In [45], by searching a Quadratic Lyapunov Functions (QLF), the robust 2-D consensus with nonlinear dynamics is guaranteed with time-invariant parametric uncertainties restricted in a polytope. In [64], also by using the method of QLF, robust consensus conditions are proposed for uncertain system where system control gains are perturbed by time-varying uncertainties with square integrable bound . In [46], contraction theory is introduced and applied to complex network with polynomial nonlinearity and time-

invariant uncertainty. Robust stability condition for this kind of system is proposed by searching a parameter-independent polynomial contraction matrix.

1.3 Mathematical Preliminaries

1.3.1 Algebraic Graph Theory

In graph theory, a weighted directed graph $\mathscr{G} = (\mathscr{A}, \mathscr{E}, G)$ of order N can be described by a set of nodes $\mathscr{A} = \{A_1, ..., A_N\}$, a set of directed edges $\mathscr{E}$ belonging to $\mathscr{A} \times \mathscr{A}$ and a weighted adjacency matrix $G = (G_{ij})_{N\times N}$. Provided that an information can be transmitted from the j-th node to the i-th node, a directed edge $e_{ij} \in \mathscr{E}$ is denoted, i.e. a directed edge $e_{ij} \in \mathscr{E}$ if and only if $G_{ij} \neq 0$. Meanwhile, A_j is called parent node and A_i is called child node. G is denoted to be positive if $G_{ij} > 0$ for all i, j, otherwise G is said to be non-positive. It is useful to give following definitions.

Definition 1.3 *(Directed path and simple path) A directed path from node A_i to A_j is a sequence of directed edges (A_i, A_{i1}),(A_{i1}, A_{i2}),...,(A_{il}, A_j). A path is called a simple path if it has no repeated vertices.*

Definition 1.4 *(Strongly connected graph) A directed network $\mathscr{G}$ is strongly connected if there is a directed path between any pair of distinct notes A_i and A_j, $i, j = 1, ..., n$, then $\mathscr{G}$ is denoted as a strongly connected graph.*

Definition 1.5 *(Reducible matrix) In a directed network $\mathscr{G}$, a matrix G is reducible if there exists a permutation matrix $P \in \mathbb{R}^N$ and an integer m with $1 \leq m \leq N-1$ satisfying*

$$P^T G P = \begin{pmatrix} \widehat{G}_{11} & 0 \\ \widehat{G}_{21} & \widehat{G}_{22} \end{pmatrix}.$$

where $\widehat{G}_{11} \in \mathbb{R}^{m\times m}$, $\widehat{G}_{21} \in \mathbb{R}^{(n-m)\times m}$ and $\widehat{G}_{22} \in \mathbb{R}^{(n-m)\times(n-m)}$. Otherwise, G is an irreducible matrix.

Definition 1.6 *An undirected graph is defined to be a tree if for any two vertices in this graph are connected by exactly one simple path. A directed tree is a directed graph if the underlying graph is a tree provided that the direction of network is ignored. A directed graph is defined to be a directed rooted tree if at least one root r has the property that, for any node v different from r, there is a unique directed path from r to v. A spanning tree is a directed rooted tree containing all the nodes of $\mathscr{G}$.*

We say that a graph contains a spanning tree if a subset of the graph forms a spanning tree. From the coupling matrix $(G_{ij})_{N\times N}$, we can obtain a Laplacian matrix defined by

$$\begin{aligned} L_{ij} &= -G_{ij}, \ \forall i \neq j \\ L_{ii} &= -\textstyle\sum_{j=1,\ j\neq i}^{n} L_{ij}. \end{aligned} \tag{1.1}$$

It is worthy to note that the the Laplacian matrix satisfies the diffusion property that

$$\sum_{j=1}^{n} L_{ij} = 0 \ \ \forall i = 1, \ldots, n. \tag{1.2}$$

Thus, 1_N is a right vector of Laplacian matrix corresponding to eigenvalue 0.

Lemma 1.1 *A matrix G is an irreducible matrix if and only if its corresponding directed graph $\mathscr{G}$ is strongly connected.*

Lemma 1.2 *The directed graph $\mathscr{G}$ has a spanning tree if and only if the Laplacian matrix L has a simple eigenvalue 0 and all the other eigenvalues are in the open right plane (have positive real parts).*

More details and applications of algebraic graph theory can be found in [66–68].

1.3.2 Lyapunov Stability Theory

In this subsection, the tools of Lyapunov stability theory will be reviewed. Firstly, basic definitions will be introduced and stability theorems will be provided later.

Consider a dynamical system satisfying

$$\dot{x} = f(x,t),\ x(t_0) = 0,\ , x \in \mathbb{R}^n \tag{1.3}$$

where standard conditions are satisfied for $f(x,t)$ with the existence and uniqueness of solutions. Then, the definition of stability in the sense of Lyapunov and asymptotic stability is given as follows

Definition 1.7 *The equilibrium point $x^* = 0$ of* (1.3) *is said to be stable (in the sense of Lypunov) at $t = t_0$ if for any $\epsilon > 0$ there is a $\delta(t_0, \epsilon) > 0$ such that*

$$\|x(t_0)\| < \delta \implies \lim_{t\to\infty} x(t) < \epsilon. \tag{1.4}$$

Stability in the sense of Lyapunov is a very mild requirement in that it just requires the trajectories starting close to the origin and remaining close to the origin. The equilibrium point is called *uniformly stable* if $\delta(t_0, \epsilon) > 0$ in above definition is independent of t_0. In other words, condition (1.4) holds for all t_0

Definition 1.8 *An equilibrium point $x^* = 0$ of* (1.3) *is asymptotically stable at $t = t_0$ if following two conditions are satisfied:*

- $x^* = 0$ *is stable.*
- $x^* = 0$ *is locally attarctive, i.e., there is a $\delta(t_0)$ such that*

$$\|x(t_0)\| < \delta \implies \lim_{t\to\infty} x(t) = 0$$

Based on Definition 1.8, uniformly asymptotic stability can be given where δ is not a function of t_0 with asymptotical stability satisfied. We say that an equilibrium point is *globally stable* if it is stable for all initial conditions $x_0 \in \mathbb{R}^n$. An equilibrium point is defined to be *unstable* if it is not stable.

Definition 1.9 *An equilibrium point is said to be an exponentially stable if there are constants* κ, $\alpha > 0$ *and* $\epsilon > 0$ *such that*

$$\|x(t)\| \leq \kappa e^{-\alpha(t-t_0)}\|x(t_0)\|,\ \forall \|x(t_0)\| \leq \epsilon,\ t \geq t_0.$$

where α *is defined to be the rate of convergence.*

Exponential stability is a strong form of stability, implying uniform and asymptotical stability.

Lyapunov direct method is proven to be an effective method which allows us to establish the stability of a system. Define B_ϵ a ball of size ϵ around the origin:

$$B_\epsilon = \{x \in \mathbb{R}^n : \|x\| \leq \epsilon\}$$

Definition 1.10 *A continuous function* $V : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}$ *is a locally positive definite function (LPDF) if there exist some* ϵ *and a continuous and strictly increasing function* $\alpha : \mathbb{R}_+ \to \mathbb{R}$ *such that*

$$V(0,t) = 0,\ V(x,t) \geq \alpha(\|x\|),\ \forall x \in B_\epsilon,\ \forall t \geq 0.$$

Definition 1.11 *A continuous function* $V : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}$ *is a positive definite function (PDF) if it is a LPDF and satisfies*

$$\lim_{s\to\infty} \alpha(s) \to \infty.$$

Definition 1.12 *A continuous function* $V : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}$ *is called a decrescent function (DSF) if there exist some* ϵ *and a continuous and strictly increasing func-*

tion $\beta : \mathbb{R}_+ \to \mathbb{R}$ *such that*

$$V(x,t) \leq \beta(\|x\|),\ \forall x \in B_\epsilon,\ \forall t \geq 0.$$

The time derivative of function V is obtained along the trajectories of the system:

$$\dot{V}|_{\dot{x}=f(x,t)} = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f.$$

In what follows, we use $\dot{V}$ instead of $\dot{V}|_{\dot{x}=f(x,t)}$. Basic Lyapunov theorems can be displayed as follows.

Lemma 1.3 *Let* $\dot{V}$ *be the derivative of* $V(x,t)$ *along the trajectories of the system, then*

- *The origin of the system is locally stable if* V *is a LPDF and* $\dot{V} \leq 0$ *locally in x for all* t*.*
- *The origin of the system is uniformly locally stable if* V *is a LPDF and is also a DSF, and* $\dot{V} \leq 0$ *locally in x for all* t*.*
- *The origin of the system is uniformly locally asymptotically stable if* V *is a LPDF and is also a DSF, and* $-\dot{V}$ *is a LPDF for all* t*.*
- *The origin of the system is uniformly globally asymptotically stable if* V *is a LPDF and is also a DSF, and* $-\dot{V}$ *is a PDF for all* t*.*

This result does not give an explicit rate of convergence according to the solutions of equilibrium. Thus, it is useful to be modified in the case of exponential stability as follows.

Lemma 1.4 *The origin is an exponentially stable equilibrium point if and only if*

there are some positive constants α_1, α_2, α_3, α_4 *and a* $\epsilon > 0$ *such that*

$$\begin{aligned} \alpha_1\|x\| \leq V(x,t) \leq \alpha_2\|x\|, \\ \dot{V} \leq -\alpha_3\|x\|, \\ \|\frac{\partial V}{\partial x}(x,t)\| \leq \alpha_4\|x\| \end{aligned}$$

where $\|x\| \leq \epsilon$.

More details can be found in standard texts as [69–72].

1.3.3 Square Matrix Representation

Whether a polynomial is semi-positive can be established effectively by determining whether it is a SOS polynomial via an LMI feasibility test. Indeed, let $f(\theta)$ be a polynomial of degree $2d_\theta$ in $\theta \in \mathbb{R}^a$. Then, $f(\theta)$ can be expressed as

$$f(\theta) = (*)^T(F + L(\delta))\phi_{\text{pol}}(\theta, d_\theta) \tag{1.5}$$

where $\phi_{\text{pol}}(\theta, d_\theta) \in \mathbb{R}^{l_{\text{pol}}(a,d_\theta)}$ is called power vector containing all monomials of degree less or equal to d_θ,

$$l_{\text{pol}}(a, d_\theta) = \frac{(a + d_\theta)!}{a!d_\theta!}. \tag{1.6}$$

$L(\delta)$ is a linear parameterization of the affine space

$$\mathscr{L}_{\text{pol}} = \{L = L^T : \ (*)^T L(\delta)\phi_{\text{pol}}(\theta, d_\theta) = 0, \ \forall\theta \in \mathbb{R}^a\} \tag{1.7}$$

in which $\delta \in \mathbb{R}^{\mu_{\text{pol}}(a,d_\theta)}$ is a vector of free parameters whose length is given by

$$\mu_{\text{pol}}(a, d_\theta) = \frac{1}{2}l_{\text{pol}}(a, d_\theta)(l_{\text{pol}}(a, d_\theta) + 1) - l_{\text{pol}}(a, 2d_\theta). \tag{1.8}$$

The representation (1.5) is known as Gram matrix method and Square Matrix Representation (SMR) [73]. This technique allows one to determine whether poly-

nomial $f(\theta)$ is a SOS. In particular, $f(\theta)$ is SOS if and only if there exists a δ such that

$$F + L(\delta) \geq 0 \tag{1.9}$$

which is a LMI feasibility test, and hence it is a convex optimization problem.

Next we will discuss the case of homogeneous polynomial. Particularly, $f(\theta)$ is a homogeneous polynomial of degree $2d_\theta$ in $\theta \in \mathbb{R}^a$. Then, $f(\theta)$ can be expressed as

$$f(\theta) = (*)^T (F + L(\delta))\phi_{\text{hom}}(\theta, d_\theta) \tag{1.10}$$

where $\phi_{\text{hom}}(\theta, d_\theta) \in \mathbb{R}^{l_{\text{hom}}(a,d_\theta)}$ is a power vector containing all homogeneous monomials of degree d_θ,

$$l_{\text{hom}}(a, d_\theta) = \frac{(a + d_\theta - 1)!}{(a-1)!d_\theta!}. \tag{1.11}$$

$L(\delta)$ is a linear parameterization of the affine space

$$\mathscr{L}_{\text{hom}} = \{L = L^T : \ (*)^T L(\delta)\phi_{\text{hom}}(\theta, d_\theta) = 0, \ \forall \theta \in \mathbb{R}^a\} \tag{1.12}$$

in which $\delta \in \mathbb{R}^{\mu_{\text{hom}}(a,d_\theta)}$ is a vector of free parameters whose length is given by

$$\mu_{\text{hom}}(a, d_\theta) = \frac{1}{2} l_{\text{hom}}(a, d_\theta)(l_{\text{hom}}(a, d_\theta) + 1) - l_{\text{hom}}(a, 2d_\theta). \tag{1.13}$$

A simple example is given here to illustrate the SMR technique. Consider polynomial $f(x) = 2x^4 + 3x^2 + 4x + 5$, one has $d_x = 2$, $a = 1$. Then, $f(x)$ can be written as follows

$$\phi_{\text{pol}}(\theta, d_x) = \begin{pmatrix} x^2 \\ x \\ 1 \end{pmatrix}, \ F = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 2 \\ 0 & 2 & 5 \end{pmatrix}, C(\delta) = \begin{pmatrix} 0 & 0 & -\delta \\ 0 & 2\delta & 0 \\ -\delta & 0 & 0 \end{pmatrix}.$$

This method can also extended to matrix polynomials. Let $M(\theta) \in \mathbb{R}^{n \times n}$ be

a matrix polynomial of degree $2d_\theta$ in $\theta \in \mathbb{R}^a$. Then, one can express $M(\theta)$ in the following form

$$M(\theta) = \Phi(\bar{M} + N(\delta), \phi_{\rm pol}(\theta, d_\theta), n) \tag{1.14}$$

and $\Phi(\bar{M} + N(\delta), \phi_{\rm pol}(\theta, d_\theta), n)$ denotes the expression

$$\Phi(\bar{M} + N(\delta), \phi_{\rm pol}(\theta, d_\theta), n) = (*)^T(M + N(\delta))(\phi_{\rm pol}(\theta, d_\theta) \otimes I_n) \tag{1.15}$$

where $\bar{M} \in \mathbb{R}^{nl_{\rm pol}(a,d_\theta)\times nl_{\rm pol}(a,d_\theta)}$ is a suitable matrix and $N(\delta)$ is the linear parameterization of the affine space

$$\mathscr{N} = \{N = N^T \in \mathbb{R}^{nl_{\rm pol}(a,d_\theta)\times nl_{\rm pol}(a,d_\theta)} : \Phi\big(N(\delta), \phi_{\rm pol}(\theta, d_\theta), n\big) = 0\} \tag{1.16}$$

whose dimension is

$$\mu_{\rm POL}(a, d_\theta, n) = \frac{1}{2}n(l_{\rm pol}(a, d_\theta)((nl_{\rm pol}(a, d_\theta) + 1) - (n + 1)l_{\rm pol}(a, 2d_\theta))). \tag{1.17}$$

The expression in the form (1.14) is used to establish whether a matrix polynomial is SOS via an LMI feasibility test. Indeed, $M(\theta)$ is a SOS if there are matrix polynomials $M_1(\theta), M_2(\theta), \ldots$ such that

$$M(\theta) = \sum_i M_i(\theta)^T M_i(\theta) \tag{1.18}$$

and this condition holds if and only if there is a δ such that the following condition holds:

$$\bar{M} + N(\delta) \geq 0. \tag{1.19}$$

Similar results can be obtained for homogeneous polynomial matrix by using (1.10). We omit it here.

It is useful to note that SOS polynomials have been investigated in optimization over polynomials for a long time. The survey [74] and references therein provide more details and algorithms about SOS polynomials.

1.4 Problem Statements

This thesis is concerned with the consensus problem of MASs with topological uncertainty. Since autonomous agent can be simply described in linear dynamics and the communication between agent is affected by noise perturbations, environmental fluctuations and parametric variations, it is natural to give a model of uncertain MASs considering these uncertainties. Furthermore, based on the matrix theory and graph theory, the consensus problem of MASs with linear dynamics has already been solved. A topological condition is proposed that consensus with fixed topology can be achieved if and only if the directed graph associated with MAS has a spanning tree. Nevertheless, it is still an open question that under what kind of condition robust consensus can be achieved with the topological uncertainties. In addition, Lyapunov stability theory, especially quadratic Lyapunov function, is widely used to construct robust consensus conditions and a number of results have been proposed. However, quadratic Lyapunov function generates conservatisms in many ways. In order to decrease the conservatism, one useful way is to construct more complex Lyapunov function with higher degrees. Thus, another question arises naturally what is the maximum degree of Lyapunov function for solving robust consensus problem. Besides that, Lyapunov direct method also requires an error dynamics which can not easily be obtained for MASs with nonlinear dynamics, additionally increasing the conservatism by using this widely-adopted method. This stimulates the researchers to implement other advanced methods for less-conservative results. Finally, polytopic parametric uncertainty is a typical model for topological disturbs. A key problem for this kind of uncertainty is to calculate the largest bound where the robust consensability remains, giving special academic interests to the robust consensus margin for topological uncertainty.

Indeed, following research problems are of great interests.

- For MASs with linear dynamics and topological uncertainties, construct robust consensus conditions both for first-order consensus and for second-order consensus.

- For robust consensus problem with nonlinear dynamics, establish both local and global consensus conditions under which the local and global robust consensus can be achieved respectively.

- For robust consensus with nonlinear dynamics, provide a more advanced method relaxed from global Lipchitz condition and without using the error-dynamics, and reduce the conservatism comparing with quadratic Lyapunov method.

- Given an uncertain MAS with polytopic uncertainties, compute the upper bound of uncertainty where the robust consensability remains, and give a solvable condition for calculating this problem.

- For nonlinear inequality conditions for robust consensus, take advantages of SMR technique, parameterise corresponding affine space and convert these problems to convex optimization problems.

1.5 Contributions

This section briefly summarizes the main contributions of this thesis as follows:

- **Robust Consensus**

  Robust consensus conditions are provided for the uncertain MASs with various consensus protocols. Specifically, the topological uncertainties are assumed to exist in the communication network, which make it difficult to check the consensability. By using parameter-dependent Lyapunov functions, both for first-order and second-order consensus, robust condition is provided respectively. Moreover, consensus problems with continuous-time and discrete-time dynamics are investigated and corresponding robust consensus condition is given respectively. With regards to these consensus conditions with topological uncertainties, solvable conditions built by employing the SMR technique are constructed and original problems can be transformed to some

optimization problems in terms of LMIs, which can be verified by standard softwares.

- **Degree Bound of Lyapunov Function**

  For robust consensus problem of MASs with linear dynamics and topological uncertainties, consensability can be guaranteed by exploiting a polynomial parameter-dependent Lyapunov function. Necessary and sufficient conditions can be provided to ensure the consensability, and the upper bound degree of Lyapunov function can be given both for first-order consensus and for second-order consensus. It shows that this upper bound is not only related to the degree of coupling matrix, but also related to the number of agents, which means for a large-scale system, a comparatively higher-degree Lyapunov function can be constructed to ensure the consensability of MAS and decrease the conservatism with respect to the results generated by quadratic Lyapunov functions. Similarly, upper bound degree of Lyapunov function could also be found for consensus problem of MASs with nonlinear dynamics, but only sufficient condition can be provided currently.

- **Robust Uncertainty Margin** Robust consensus problems can be solved by using Lyapunov stability theory, and robust consensability can be guaranteed with regards to topological uncertainties constrained in a certain semi-algebraic set or a polytope. Firstly, by exploiting a HPLF the robust local consensus problem of MASs with nonlinear dynamics can be transformed to a robust stability problem. Using SMR technique, the robust consensus margin can be calculated by solving a GEVP. On the other hand, by using contraction theory, the robust global consensus can also be ensured via constructing a parameter-dependent contraction matrix. Based on this robust consensus condition, the maximum polytopic uncertainty can be checked for globally asymptotically consensus, which is a convex optimization problem consisting of GEVPs. By using this margin, a comparison between proposed

method with quadratic Lyapunov function and a parameter-independent contraction matrix can be launched.

- **Contraction Analysis**

  Contraction theory is introduced in robust analysis of consensus control, and a new type of contraction matrix, the homogeneous parameter-dependent polynomial contraction matrix, is built in order to handle robust consensus problem of MASs with nonlinear dynamics. Indeed, a parameter-dependent contraction matrix is introduced and robust consensus condition is proposed for MASs with topological uncertainties via using the partial contraction and constructing an auxiliary system. Furthermore, with regards to the polynomial nonlinearity, HPD-PCM is established to ensure the robust consensus with topological uncertainties. It shows that this new type of contraction matrix displays obvious advantages comparing to affine parameter-dependent quadratic Lyapunov function and parameter-independent polynomial contraction matrix.

1.6 Thesis Outline

This thesis is organised as follows:

- **Chapter 2** investigates robust consensus for uncertain MASs with linear dynamics. Specifically, it is assumed that the communications of network is described by a weighted adjacency matrix whose entries are generic polynomial functions of a vector constrained in a set depicted by generic polynomial inequalities. For this uncertain structure, necessary and sufficient conditions are given to guarantee the robust first-order and robust second-order consensus, also in both cases of positive and non-positive couplings. On the other hand, robust consensus is also investigated for MASs with discrete-time dynamics. Both considering systems with single and double integrators, necessary and sufficient conditions are proposed for robust consensus based on

the existence of a polynomial parameter-dependent Lyapunov function. In particular, for achieving necessity, an upper bound on the required degree of Lyapunov function is given. Based on the zeros of a polynomial, a necessary and sufficient condition is given for ensuring the robust consensus with single integrator and nonnegative coupling matrices .

- **Chapter 3** considers local and global consensus of MASs with nonlinear dynamics, regarding to limited solution manifold. For local consensus, the original system is firstly transformed to a polytopic system. Then, a sufficient condition is provided based on the use of HPLF. For global consensus, another method is provided via searching for a proper PLF. Also, the proposed methods use more complex Lyapunov functions than QLFs which can be deemed as a special case of proposed method. In addition, this chapter also investigates robust local consensus in MASs with time-varying topological uncertainties constrained in a polytope. In contrast to traditional methods with non-convex conditions via using QLF, a novel criteria is given based on using HPLFs where the original system is properly approximated by an uncertain polytopic system. Moreover, corresponding solvable conditions in terms of LMIs have been given by using SMR technique. Then, polytopic consensus margin problem is introduced and investigated via tackling with GEVPs.

- **Chapter 4** concerns robust global consensus problem of polynomial system disturbed by time-varying uncertainties on topology. In particular, structured parameters is assumed to be in a bounded-rate polytope. Novel consensus conditions are proposed by using parameter-dependent contraction matrices for robust exponential consensus and for robust asymptotical consensus. Moreover, for polynomial intrinsic function, by introducing a new class of contraction matrix, i.e., HPD-PCM, tractable conditions in terms of LMIs are provided via some suitable affine space parameterizations. In addition, the variant rate margin for robust asymptotical consensus is given by han-

dling GEVPs. Moore-Greitzer jet engines and a six agent system are used to demonstrate the effectiveness of proposed methods.

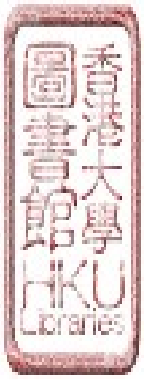
- **Chapter 5** concludes this thesis by showing final remarks and pointing out our future works.

Chapter 2

Robust Consensus for Uncertain and Linear Dynamics

2.1 Introduction

This chapter considers MASs with linear dynamics and topological uncertainties, both for continuous-time consensus protocol and for discrete-time consensus protocol. Specifically, it is under the assumption that the entries of weighted adjacency matrix are generic polynomial functions of a vector of uncertain parameter constrained in a semi-algebraic set. Firstly, necessary and sufficient conditions are provided to ensure the robust first-order and robust second-order consensus for continuous-time consensus protocols, and for both cases of positive and non-positive weighted adjacency matrices. Moreover, it shows how these conditions can be transformed and checked by convex optimization programming. On the other hand, robust consensus problem with discrete-time dynamics is also investigated. Necessary and sufficient conditions are given for robust consensus based on finding a polynomial parameter-dependent Lyapunov function. In order to achieve necessity, it is proposed an upper bound on the degree that candidate Lyapunov function requires. Through checking the zeros of a constructed polynomial, a necessary and sufficient condition is provided for robust first-order consensus with nonnegative



network weights. Lastly, it is also shown how these conditions can be transformed to convex optimization problems by SOS technique. Numerical examples illustrate the usefulness of proposed methods.

This chapter is organized as follows. Section 2.2 formulates the problems. Section 2.3 proposes conditions both for robust first-order and robust second-order consensus of MASs with continuous-time protocols. With regards to discrete-time protocols, Section 2.4 proposes consensus conditions both for robust first-order and robust second-order consensus. Section 2.5 illustrates the proposed methods with numeral examples. Section 2.6 concludes this chapter with some final remarks.

2.2 Problem Formulation

In this chapter, uncertain MASs are considered where the weighted adjacency matrix is disturbed by an uncertain vector. Let such a matrix be $G(\theta) \in \mathbb{R}^{N\times N}$ where N is the number of agents of MAS and $\theta \in \mathbb{R}^a$ is an uncertain vector satisfying

$$\theta \in \Omega \tag{2.1}$$

where

$$\Omega = \{\theta \in \mathbb{R}^a : \ s_i(\theta) \geq 0 \, \forall i = 1, \ldots, h\} \tag{2.2}$$

for some functions $s_1, \ldots, s_h : \mathbb{R}^a \to \mathbb{R}$. Therein after we will assume that the entries of $G(\theta)$ and $s_1(\theta), \ldots, s_h(\theta)$ are polynomial functions. In addition, we denote that $G(\theta)$ is positive if $G_{ij}(\theta) > 0$ for all i, j and for all $\theta \in \Omega$, otherwise $G(\theta)$ is said to be non-positive.

For robust first-order consensus, we concern with the continuous-time uncertain MASs described by

$$\dot{x}_i(t) = \sum_{j=1,\ j\neq i}^{N} G_{ij}(\theta)(x_j(t) - x_i(t)), \ \ i = 1, \ldots, N \tag{2.3}$$

where x_i is the state of the i-th node, and $G(\theta)$ could be either positive or non-

positive. Thus, here we can propose the robust first-order consensus problem as follows.

Problem 2.1 *To establish if the uncertain MASs* (2.3) *achieves robust first-order consensus for all uncertain parameters and for any initial states, i.e.*

$$\lim_{t\to\infty} x_i(t) - x_j(t) = 0 \ \ \forall i, j \ \forall \theta \in \Omega. \tag{2.4}$$

Aims to address above problem, we would like to rewrite the uncertain MASs (2.3) as

$$\dot{x}(t) = -L(\theta)x(t) \tag{2.5}$$

where $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ is called the state vector, and $L(\theta) = (L_{ij}(\theta))_{n\times n}$ is said the uncertain Laplacian matrix satisfying

$$\begin{array}{rcl} L_{ij}(\theta) & = & -G_{ij}(\theta) \ \forall i \neq j \\ L_{ii}(\theta) & = & -\sum_{j=1,\ j\neq i}^{n} L_{ij}(\theta). \end{array} \tag{2.6}$$

It is worthy to point out that the uncertain Laplacian matrix defined above has the diffusion property that

$$\sum_{j=1}^{n} L_{ij}(\theta) = 0 \ \ \forall i = 1, \ldots, n. \tag{2.7}$$

For robust second-order consensus problem, the following continuous-time uncertain MASs is considered

$$\begin{array}{rcl} \dot{x}_i(t) & = & \rho_i(t) \\ \dot{\rho}_i(t) & = & \displaystyle\sum_{j=1,\ j\neq i}^{n} \alpha G_{ij}(\theta)(x_j(t) - x_i(t)) \\ & & \displaystyle + \sum_{j=1,\ j\neq i}^{n} \beta G_{ij}(\theta)(\rho_j(t) - \rho_i(t)) \end{array} \tag{2.8}$$

in which $x_i \in \mathbb{R}$ is the position state of the i-th node, $\rho_i \in \mathbb{R}$ is the velocity state of the i-th node, and $\alpha, \beta \in \mathbb{R}$ are constants. We also denote that $G(\theta)$ in this case can

be either positive or non-positive matrix. Thus, the robust second-order consensus problem can be proposed as follows.

Problem 2.2 *To establish if the uncertain MASs* (2.8) *achieves robust second-order consensus for any initial state, i.e.*

$$\begin{array}{rcl} \lim\limits_{t\to\infty} x_i(t) - x_j(t) & = & 0 \\ \lim\limits_{t\to\infty} \rho_i(t) - \rho_j(t) & = & 0 \end{array} \quad \forall i, j \; \forall \theta \in \Omega. \tag{2.9}$$

For the sake of addressing this problem, we would like to rewrite the uncertain MASs (2.8) as

$$\begin{array}{rcl} \dot{x}_i(t) & = & \rho_i(t) \\ \dot{\rho}_i(t) & = & -\sum\limits_{j=1}^{n} \alpha L_{ij}(\theta) x_j(t) - \sum\limits_{j=1}^{n} \beta L_{ij}(\theta) \rho_j(t) \end{array} \tag{2.10}$$

where $x \in \mathbb{R}^n$ is the position state vector and $\rho \in \mathbb{R}^n$ is the velocity state vector. We define the global state vector as $y = (x^T, \rho^T)^T \in \mathbb{R}^{2n}$. Then, a compact form of system (2.10) can be intrduced as

$$\dot{y}(t) = \widetilde{L}(\theta) y(t) \tag{2.11}$$

where $\widetilde{L}(\theta)$ is called the uncertain extended Laplacian matrix provided by

$$\widetilde{L}(\theta) = \begin{bmatrix} 0 & I \\ -\alpha L(\theta) & -\beta L(\theta) \end{bmatrix}. \tag{2.12}$$

2.3 Continuous-time Dynamics

This section provides both the robust first-order and second-order consensus conditions.

2.3.1 First-order Consensus

Lyapunov stability theory is broadly adopted to investigate the properties of dynamical system. By associating the robust consensus with Lyapunov stability theory, we propose a new condition for establishing robust first-order consensus based on LMIs. Specifically, first we define a matrix $V_1 \in \mathbb{R}^{n\times n-1}$ satisfying

$$\text{img}(V_1) = \text{ker}(1_n^T) \tag{2.13}$$

where 1_n is a column vector of dimension n with every entry being 1.

Then the transformed uncertain Laplacian matrix is introduced as:

$$\widehat{L}(\theta) = V_1^T L(\theta) V_1. \tag{2.14}$$

Theorem 2.1 *With either positive or non-positive weights, robust first-order consensus for uncertain MAS can be obtained if and only if there exists a polynomial matrix $P_1 : \mathbb{R}^a \to \mathbb{R}^{n-1\times n-1}$ such that*

$$\begin{cases} P_1(\theta) > 0 \\ P_1(\theta)\widehat{L}(\theta) + \widehat{L}(\theta)^T P_1(\theta) > 0 \end{cases} \quad \forall \theta \in \Omega. \tag{2.15}$$

Proof Let us observe that 1_n is an eigenvector of $L(\theta)$ corresponding to the eigenvalue zero. In addition, one can also observe that $V_1^T L(\theta) V_1$ has the same eigenvalues of $L(\theta)$ while the algebraic multiplicity of the eigenvalue zero has been reduced by one. i.e.

$$\text{spc}(\widehat{L}(\theta)) \cup \{0\} = \text{spc}(L(\theta)) \tag{2.16}$$

Define a dynamical system as follows.

$$\dot{\hat{x}}(t) = -\widehat{L}(\theta)\hat{x}(t). \tag{2.17}$$

One can observe that $x = \gamma 1_n$ is the equilibrium point of (2.17),$\forall \gamma \in \mathbb{R}$. Thus (2.17) is asymptotically stable is equivalent to the statement that the robust first-order consensus can be obtained. With respect to (2.16) and the Lyapunov stability theorem, one necessary and sufficient condition that (2.17) is asymptotically stable for all $\theta \in \Omega$ is that $L(\theta)$ has exactly one simple eigenvalue 0 and all the other eigenvalues locate in the open right plane. Based on Lyapunov stability theorem for linear systems, this is equivalent to saying that there is a $P_1(\theta)$ such that (2.15) holds for all $\theta \in \Omega$. Thus, this theorem holds $\square$

In order to check the condition of Theorem 2.1, one useful way is to exploit SOS matrix polynomials. Indeed, let $P_1(\theta)$ and $G_{1i}(\theta)$, $i = 1, \ldots, h$, be symmetric matrix polynomials to be determined, and let

$$R_1(\theta) = P_1(\theta)\widehat{L}(\theta) + \widehat{L}(\theta)^T P_1(\theta) - \sum_{i=1}^{h} G_{1i}(\theta) s_{1i}(\theta). \tag{2.18}$$

It is obvious to obtain that (2.15) holds if there exists $c > 0$ such that

$$\begin{cases} G_{1i}(\theta) \text{ is SOS} \\ P_1(\theta) - I_{n-1} \text{ is SOS} \\ R_1(\theta) - cI_{n-1} \text{ is SOS}. \end{cases} \tag{2.19}$$

Actually, as long as the constraints in (2.19) hold with $c > 0$, for any $\theta \in \Omega$, it directly follows that $G_{1i}(\theta) \geq 0$, $P_1(\theta) > 0$ and

$$\begin{aligned} 0 &\leq P_1(\theta)\widehat{L}(\theta) + \widehat{L}(\theta)^T P_1(\theta) - \sum_{i=1}^{h} G_{1i}(\theta) s_{1i}(\theta) - cI_{n-1} \\ &\leq P_1(\theta)\widehat{L}(\theta) + \widehat{L}(\theta)^T P_1(\theta) - cI_{n-1} \\ &\leq P_1(\theta)\widehat{L}(\theta) + \widehat{L}(\theta)^T P_1(\theta) \end{aligned} \tag{2.20}$$

i.e. (2.15) holds.

The condition (2.19) can be established via a convex optimization problem with LMI constraints by exploiting the representation of matrix polynomials reported in Section 1.3.3. Specifically, let $2m_i$ be the degree of $G_{1i}(\theta)$, $2m$ be the degree of

$P_1(\theta)$, and $2m_0$ be the degree of $R_1(\theta) - cI$. Here we introduce the representations

$$\begin{aligned} G_{1i}(\theta) &= \Phi(\bar{G}_{1i}, \phi_{\text{pol}}(\theta, m_i), n-1) \\ G_{1i}(\theta)s_{1i}(\theta) &= \Phi(\bar{U}_{1i}(\bar{G}_{1i}), \phi_{\text{pol}}(\theta, m_0), n-1) \\ P_1(\theta) &= \Phi(\bar{P}_1, \phi_{\text{pol}}(\theta, m), n-1) \\ R_1(\theta) &= \Phi(\bar{F}_1 + D_1(\delta), \phi_{\text{pol}}(\theta, m_0), n-1) \end{aligned} \tag{2.21}$$

where $\bar{G}_{1i}$, $\bar{U}_{1i}(\bar{G}_{1i})$, $\bar{P}_1$, $\bar{F}_1$ and $D_1(\delta)$ are symmetric matrices. Moreover, let us define

$$\begin{aligned} & c^* = \sup_{c, \bar{G}_{1i}, \bar{P}_1, \delta} c \\ & \text{s.t.} \begin{cases} \bar{G}_{i1} \geq 0 \\ \bar{P}_1 \geq I_{s_1} \\ \bar{F}_1 + D_1(\delta) - cI_{s_2} - \sum_{i=1}^{h} \bar{U}_{1i}(\bar{G}_{1i}) \geq 0 \end{cases} \end{aligned} \tag{2.22}$$

in which s_1 and s_2 give the sizes of $\bar{P}_1$ and $\bar{F}_1$, respectively. Then, it follows that (2.15) holds if $c^* > 0$.

Remark 2.1 *It is worth noting the relationship between conditions* (2.15), (2.19) *and* (2.22). *One can use condition* (2.19) *to check the feasibility of* (2.15), *while* (2.19) *is only a sufficient condition for robust first-order consensus. In addition, condition* (2.19) *in terms of SOS is equivalent to the LMI condition* (2.22).

For a network topology with positive coupling weights but without parametric uncertainties, it has been proved that the structure of network plays a key role to determines whether the consensus can be obtained. The following result extends to the case of uncertain MASs existing conditions obtained for the case of MASs without uncertainty [15], and gives a further condition based on zeros of a polynomial.

Theorem 2.2 *For a given* $L(\theta)$ *in* (2.6) *and a directed network* $\mathscr{G} = (\mathscr{A}, \mathscr{E}, G(\theta))$ *with a positive weights, i.e.* $\exists e_{ij} \in \mathscr{E}$ *if and only if* $G_{ij}(\theta) > 0$, *the statements as follows are equivalent.*

a) Robust first-order consensus can be achieved.

b) $\forall\theta \in \Omega$, $L(\theta)$ *has exactly one simple eigenvalue* 0 *and all the other eigenvalues locate in the open right half plane.*

c) $\forall\theta \in \Omega$, *the directed network* $\mathcal{G}$ *has a spanning tree.*

d) $\forall\theta \in \Omega$, $q(\theta) \neq 0$, *where*

$$q(\theta) = \frac{d}{d\lambda} l(\lambda, \theta)\Big|_{\lambda=0} \tag{2.23}$$

and

$$l(\lambda, \theta) = \det(\lambda I - L(\theta)). \tag{2.24}$$

Proof Assume the Laplacian matrix $L(\theta)$ is given by (2.6). Then, the first three statements can be proved to be equivalent which follows directly from the analogous ones obtained for the case of MASs without uncertainty [15]. From Lemma 3.3 in [15], one can obtain that $\Re(\lambda_i(L(\theta))) \geq 0$, $\forall i = 1, 2..., n$, $\forall\theta \in \Omega$. In addition, statement d) implies that $L(\theta)$ has exactly one zero eigenvalue, $\forall\theta \in \Omega$. Therefore, statements b) and d) are also equivalent, which completes this proof. □

One effective way to check the condition of Theorem 2.2 consists of exploiting SOS polynomials which amounts to solving an LMI problem. Indeed, let us first define

$$c^* = \sup_{c, g_i(\theta)} c$$
$$\text{s.t.} \begin{cases} g_i(\theta) \text{ is SOS} \\ (-1)^k q(\theta) - c - \sum_{i=1}^{h} g_i(\theta) s_{2i}(\theta) \text{ is SOS} \end{cases} \tag{2.25}$$

where $k \in \{0, 1\}$ is defined as

$$k = \begin{cases} 0 & \text{if } q(\theta_0) > 0 \\ 1 & \text{otherwise} \end{cases} \tag{2.26}$$

and θ_0 is any vector θ in Ω which can be arbitrarily chosen. Then, condition of Theorem 2.2 holds if $c^* > 0$.

Indeed, it turns out that c^* is a lower bound of $q(\theta)$ for $\theta \in \Omega$ if $q(\theta_0) > 0$, otherwise c^* is a lower bound of $-q(\theta)$ for $\theta \in \Omega$. Actually, as long as the constraints in (2.25) hold, for any $\theta \in \Omega$ it follows that

$$\begin{aligned} 0 &\leq (-1)^k q(\theta) - c - \sum_{i=1}^{h} g_i(\theta) s_{2i}(\theta) \\ &\leq (-1)^k q(\theta) - c \end{aligned} \tag{2.27}$$

i.e. c is a lower bound of $(-1)^k q(\theta)$ for $\theta \in \Omega$.

c^* in (2.25) can be obtained by solving an LMI problem by using the representation of polynomials given in Subsection 1.3.3. Indeed, let $2m_i$ and $2m_0$ be the degree of $g_i(\theta)$ and $(-1)^k q(\theta) - c - \sum_{i=1}^{h} g_i(\theta) s_{2i}(\theta)$ respectively. Let us introduce the following representations

$$\begin{aligned} g_i(\theta) &= (*)^T G_{2i} \phi_{\text{pol}}(\theta, m_i), \\ g_i(\theta) s_{2i}(\theta) &= (*)^T U_{2i}(G_{2i}) \phi_{\text{pol}}(\theta, m_0), \\ (-1)^k q(\theta) &= (*)^T (F + C(\delta)) \phi_{\text{pol}}(\theta, m_0), \\ 1 &= (*)^T W \phi_{\text{pol}}(\theta, m_0), \end{aligned} \tag{2.28}$$

where G_{2i}, $U_{2i}(G_{2i})$, F, $C(\delta)$ and W are symmetric matrices. Then,

$$\begin{aligned} & c^* = \sup_{c, G_{2i}, \delta} c \\ \text{s.t.} \; & \begin{cases} G_i \geq 0 \\ F + C(\delta) - cW - \sum_{i=1}^{h} U_{2i}(G_{2i}) \geq 0. \end{cases} \end{aligned} \tag{2.29}$$

Let us remark that problem (2.29) is a convex optimization problem with LMI constraints and linear cost function, which can be considered as eigenvalue problem and semidefinite program [75].

2.3.2 Second-order Consensus

Let us investigate the uncertain expanded Laplacian matrix $\widetilde{L}(\theta)$. Firstly, we will provide the following result, which extends to the case of uncertain MASs the condition in [21] for the case where topological uncertainty are not considered.

Lemma 2.1 *For all $\theta \in \Omega$, robust second-order consensus for the uncertain MASs*(2.11) *can be achieved if and only if the uncertain expanded Laplacian matrix $-\widetilde{L}(\theta)$ has exactly one zero eigenvalue with algebraic multiplicity two and all the other eigenvalues are in the open right half plane.*

Based on this result, we propose a new condition for checking robust second-order consensus based on matrix inequalities. Specifically, let us introduce vectors as

$$u_1 = \begin{pmatrix} 1_n \\ 0_n \end{pmatrix}, \;\; u_2 = \begin{pmatrix} 0_{n-1} \\ 1_n \end{pmatrix}. \tag{2.30}$$

Let $V_2 \in \mathbb{R}^{2n \times 2n-1}$ and $V_3 \in \mathbb{R}^{2n-1 \times 2n-2}$ be matrices satisfying

$$\begin{aligned} \text{img}(V_2) &= \ker(u_1^T) \\ \text{img}(V_3) &= \ker(u_2^T). \end{aligned} \tag{2.31}$$

Then, let us introduce the transformed uncertain expanded Laplacian matrix:

$$\breve{L}(\theta) = -V_3^T V_2^T \widetilde{L}(\theta) V_2 V_3. \tag{2.32}$$

Theorem 2.3 *Robust second-order consensus for uncertain MASs with either positive or non-positive weights can be obtained if and only if there is a symmetric function $P_2 : \mathbb{R}^a \to \mathbb{R}^{2n-2 \times 2n-2}$ such that*

$$\begin{cases} P_2(\theta) > 0 \\ P_2(\theta)\breve{L}(\theta) + \breve{L}(\theta)^T P_2(\theta) > 0 \end{cases} \quad \forall \theta \in \Omega. \tag{2.33}$$

Proof One can observe that u_1 is an eigenvector of $\widetilde{L}(\theta)$ corresponding to the eigenvalue zero. In addition, one can also observe that $V_2^T\widetilde{L}(\theta)V_2$ has the same eigenvalues of $\widetilde{L}(\theta)$ while the algebraic multiplicity of the eigenvalue zero has been reduced by one. By a similar way, it follows that $V_3^TV_2^T\widetilde{L}(\theta)V_2V_3$ has the same eigenvalues of $\widetilde{L}(\theta)$ while the algebraic multiplicity of the eigenvalue zero has been reduced by two. Hence, from Lemma 2.1, it directly follows that robust second-order consensus can be obtained if and only if $-\widetilde{L}(\theta)$ has all the eigenvalues locating in the open right half plane for all $\theta \in \Omega$. From Lyapunov stability theorem for linear systems, this is equivalent to the statement that there is a $P_2(\theta)$ such that (2.33) holds for all $\theta \in \Omega$. Therefore, this proof completes. □

With the purpose of investigating the condition of Theorem 2.3, one can employ SOS matrix polynomials. Indeed, let $P_2(\theta)$ and $G_{3i}(\theta)$, $i = 1, \ldots, h$, be symmetric matrix polynomials to be determined, and let us define

$$R_2(\theta) = P_2(\theta)\breve{L}(\theta) + \breve{L}(\theta)^T P_2(\theta) - \sum_{i=1}^{h} G_{3i}(\theta)s_{3i}(\theta). \tag{2.34}$$

It is easy to verify that (2.33) holds if there exists $c > 0$ such that

$$\begin{cases} G_{3i}(\theta) \text{ is SOS} \\ P_2(\theta) - I_{2n-2} \text{ is SOS} \\ R_2(\theta) - cI_{2n-2} \text{ is SOS}. \end{cases} \tag{2.35}$$

Actually, as long as the constraints in (2.35) hold with $c > 0$, for any $\theta \in \Omega$ it directly follows that $G_{3i}(\theta) \geq 0$, $P_2(\theta) > 0$ and

$$\begin{aligned} 0 \;&\leq\; P_2(\theta)\breve{L}(\theta) + \breve{L}(\theta)^T P_2(\theta) - \sum_{i=1}^{h} G_{3i}(\theta)s_{3i}(\theta) \\ &\quad\; -cI_{2n-2} \\ &\leq\; P_2(\theta)\breve{L}(\theta) + \breve{L}(\theta)^T P_2(\theta) - cI_{2n-2} \\ &\leq\; P_2(\theta)\breve{L}(\theta) + \breve{L}(\theta)^T P_2(\theta) \end{aligned} \tag{2.36}$$

i.e. (2.33) holds.

By using the Gram Matrix Methods reported in Subsection 1.3.3, the condition (2.35) can also be formulated as a convex optimization problem in terms of LMIs. Specifically, let $2m_i$, $2m$ and $2m_0$ be the degree of $G_{3i}(\theta)$, $P_2(\theta)$ and $R_2(\theta) - cI$ respectively. Let us introduce the representations

$$\begin{aligned} G_{3i}(\theta) &= \Phi(\bar{G}_{3i}, \phi_{\text{pol}}(\theta, m_i), 2n-2) \\ G_{3i}(\theta)s_{3i}(\theta) &= \Phi(\bar{U}_{3i}(\bar{G}_{3i}), \phi_{\text{pol}}(\theta, m_0), 2n-2) \\ P_2(\theta) &= \Phi(\bar{P}_2, \phi_{\text{pol}}(\theta, m), 2n-2) \\ R_2(\theta) &= \Phi(\bar{F}_2 + D_2(\delta), \phi_{\text{pol}}(\theta, m_0), 2n-2) \end{aligned} \tag{2.37}$$

where $\bar{G}_{3i}$, $\bar{U}_{3i}(\bar{G}_{3i})$, $\bar{P}_2$, $\bar{F}_2$ and $D_2(\delta)$ are symmetric matrices. Then, define

$$\begin{aligned} c^* = & \sup_{c, \bar{G}_{3i}, \bar{P}_2, \delta} \quad c \\ \text{s.t.} & \begin{cases} \bar{G}_{3i} \geq 0 \\ \bar{P}_2 \geq I_{s_3} \\ \bar{F}_2 + D_2(\delta) - cI_{s_4} - \sum_{i=1}^{h} \bar{U}_{3i}(\bar{G}_{3i}) \geq 0 \end{cases} \end{aligned} \tag{2.38}$$

where s_3 and s_4 are the sizes of $\bar{P}_2$ and $\bar{F}_2$, respectively. Then, one can obtain that (2.33) holds if $c^* > 0$.

2.4 Discrete-time Dynamics

In this section we consider robust consensus for MASs with discrete-time dynamics affected by uncertainty, which describes the presence of unknown control gains. Specifically, for MASs with first-order dynamics we consider the following update scheme

$$x_i(k+1) = \frac{1}{\sum_{j=1}^{n} G_{ij}(\theta)} \sum_{j=1}^{n} G_{ij}(\theta) x_j(k), \ i = 1, \ldots, n \tag{2.39}$$

where $x_i \in \mathbb{R}$ is the state of the i-th node, $\theta \in \mathbb{R}^a$ is a vector of uncertain parameters, and $G : \mathbb{R}^a \to \mathbb{R}^{n\times n}$ is a generic polynomial function. The set (2.2) constrains θ. One can rewrite the system (2.39) in compact form as

$$x(k+1) = D(\theta)x(k) \tag{2.40}$$

where $D : \mathbb{R}^a \to \mathbb{R}^{n\times n}$ is given by

$$D_{ij}(\theta) = \frac{G_{ij}(\theta)}{\sum_{k=1}^{n} G_{ik}(\theta)}, \;\; i,j = 1,\ldots,n. \tag{2.41}$$

Problem 2.3 *To establish whether* (2.39) *obtains robust consensus, i.e.*

$$\lim_{k\to\infty} x_i(k) - x_j(k) = 0 \;\; \forall i,j = 1,\ldots,n \; \forall x(0) \in \mathbb{R}^n \; \forall \theta \in \Omega. \tag{2.42}$$

For MASs with double integrator, let us consider the model

$$\begin{cases} x_i(k+1) &= x_i(k) + \varrho_i(k) \\ \varrho_i(k+1) &= \varrho_i(k) + u_i(k) \end{cases} \tag{2.43}$$

with

$$u_i(k) = k_1 \sum_{j=1}^{n} G_{ij}(\theta)(x_j(k) - x_i(k)) + k_2 \sum_{j=1}^{n} G_{ij}(\theta)(\varrho_j(k) - \varrho_i(k)) \tag{2.44}$$

where $k_1, k_2 \in \mathbb{R}$ are positive scalars depicting coupling strengths, and $x_i, \varrho_i \in \mathbb{R}$ stand for the position and velocity states of the i-th agent respectively. Let us use a compact form to represent the system (2.43) as

$$\begin{pmatrix} x(k+1) \\ \varrho(k+1) \end{pmatrix} = \Gamma(\theta) \begin{pmatrix} x(k) \\ \varrho(k) \end{pmatrix} \tag{2.45}$$

where

$$\Gamma(\theta) = \begin{pmatrix} I_n & I_n \\ -k_1 L(\theta) & I_n - k_2 L(\theta) \end{pmatrix} \tag{2.46}$$

and $L(\theta) \in \mathbb{R}^{n\times n}$ is the uncertain Laplacian matrix given by

$$\begin{aligned} L_{ij}(\theta) &= -G_{ij}(\theta) \ \forall i \neq j \\ L_{ii}(\theta) &= -\sum_{j=1,\ j\neq i}^{n} L_{ij}(\theta). \end{aligned} \tag{2.47}$$

Problem 2.4 *To establish if* (2.43) *obtains robust consensus, i.e.*

$$\begin{cases} \lim\limits_{k\to\infty} x_i(k) - x_j(k) = 0 \\ \lim\limits_{k\to\infty} \varrho_i(k) - \varrho_j(k) = 0 \end{cases} \quad \forall i,j = 1,\ldots,n,\ \forall x(0), \varrho(0) \in \mathbb{R}^n\ , \forall \theta \in \Omega. \tag{2.48}$$

In the follow-up, it is assumed that $G(\theta)$ is well-posed over Ω, i.e.

$$\sum_{k=1}^{n} G_{ik}(\theta) \neq 0 \ \ \forall i = 1,\ldots,n\ \forall \theta \in \Omega. \tag{2.49}$$

Same with last section, we also denote that $G(\theta)$ is nonnegative if $G_{ij}(\theta) \geq 0$ for all $i,j = 1,\ldots,n$ and for all $\theta \in \Omega$.

It is useful to borrow the definition of (row) stochastic matrix, i.e., a nonnegative matrix with the property that all row sums are 1 [76]. One can observe that $D(\theta)$ is a stochastic matrix if $G(\theta)$ is nonnegative.

2.4.1 First-order Consensus

Let us first introduce the following polynomial

$$\zeta(\theta) = \text{LCM}\left\{\sum_{j=1}^{n} G_{ij}(\theta),\ i = 1,\ldots,n\right\}. \tag{2.50}$$

and define

$$K(\theta) = \zeta(\theta) D(\theta). \tag{2.51}$$

Let $V_1 \in \mathbb{R}^{n\times(n-1)}$ satisfy

$$\begin{cases} \text{img}(V_1) &= \ker(1_n^T) \\ V_1^T V_1 &= I_{n-1} \end{cases} \tag{2.52}$$

and define

$$D_1(\theta) = V_1^T D(\theta) V_1 \tag{2.53}$$

and

$$K_1(\theta) = V_1^T K(\theta) V_1. \tag{2.54}$$

The following result gives a necessary and sufficient condition to determine whether (2.39) obtains robust consensus.

Theorem 2.4 *Let τ be the degree of $G(\theta)$, and define*

$$\mu_1 = n(n^2 - n - 2)\tau. \tag{2.55}$$

The system (2.39) *obtains robust consensus if and only if there is a symmetric matrix polynomial $P(\theta) \in \mathbb{R}^{(n-1)\times(n-1)}$ of degree $d \leq \mu_1$ such that*

$$\begin{cases} P(\theta) > 0 \\ \zeta(\theta)^2 P(\theta) - K_1(\theta)^T P(\theta) K_1(\theta) > 0 \end{cases} \quad \forall \theta \in \Omega. \tag{2.56}$$

Proof (Sufficiency) Provided that (2.56) holds, based on Lyapunov stability theorem for discrete-time linear systems, it provides that

$$|\lambda_i (K_1(\theta))| < |\zeta(\theta)| \quad \forall i = 1, \ldots, n-1 \; \forall \theta \in \Omega \tag{2.57}$$

where $\lambda_i (K_1(\theta))$ is the i-th eigenvalue of $K_1(\theta)$. Since

$$K_1(\theta) = \zeta(\theta) D_1(\theta) \tag{2.58}$$

it hence has that $D_1(\theta)$ is Schur for all $\theta \in \Omega$, i.e.

$$|\lambda_i (D_1(\theta))| < 1 \ \ \forall i = 1, \ldots, n-1 \ \forall \theta \in \Omega. \tag{2.59}$$

As 1 is an eigenvalue of $D(\theta)$, one can rewrite the characteristic polynomial of $D(\theta)$ as

$$\det (\lambda I_n - D(\theta)) = (\lambda - 1)\xi(\lambda, \theta). \tag{2.60}$$

Since 1_n is an eigenvector of $D(\theta)$ with regard to the eigenvalue 1, it directly follows that the characteristic polynomial of $D_1(\theta)$ is provided by

$$\det (\lambda I_{n-1} - D_1(\theta)) = \xi(\lambda, \theta) \tag{2.61}$$

i.e. $D_1(\theta)$ has the same eigenvalues of $D(\theta)$ while the algebraic multiplicity of the eigenvalue 1 has been reduced by one. Hence, as $D_1(\theta)$ is Schur for all $\theta \in \Omega$, one can obtain that $D(\theta)$ has exactly one simple eigenvalue 1 and all the other eigenvalues with magnitude less than 1 for all $\theta \in \Omega$. From [15] it is equivalent to saying that consensus is obtained for all $\theta \in \Omega$.

(Necessity) Let us assume that (2.39) achieves robust consensus. From [15] this directly implies that $D(\theta)$ has exactly one simple eigenvalue 1 and all the others are with a magnitude smaller than 1 for all $\theta \in \Omega$. It is equivalent to saying that $D_1(\theta)$ is Schur for all $\theta \in \Omega$, and hence that the Lyapunov equation

$$P(\theta) - D_1(\theta)^T P(\theta) D_1(\theta) = Q(\theta) \tag{2.62}$$

has a unique solution $P(\theta)$ which satisfies $P(\theta) > 0$ for all $\theta \in \Omega$ whenever $Q(\theta) > 0$ for all $\theta \in \Omega$. Since $\zeta(\theta) \neq 0$ for all $\theta \in \Omega$, this equation can be represented as

$$\zeta(\theta)^2 P(\theta) - K_1(\theta)^T P(\theta) K_1(\theta) = \zeta(\theta)^2 Q(\theta). \tag{2.63}$$

Let us gather the $n(n-1)/2$ free entries of $P(\theta)$ and $Q(\theta)$ into the vectors $p(\theta)$ and $q(\theta)$. One can rewrite the equation above as

$$E(\theta)p(\theta) = \zeta(\theta)^2 q(\theta). \tag{2.64}$$

Since the solution $P(\theta)$ exists and it is also unique, it means that $E(\theta)$ is invertible for all $\theta \in \Omega$, and hence

$$p(\theta) = \frac{\text{adj}(E(\theta))}{\det(E(\theta))}\zeta(\theta)^2 q(\theta). \tag{2.65}$$

As the degrees of $\zeta(\theta)$ and $K_1(\theta)$ are not greater than $n\tau$, it directly follows that the degree of $E(\theta)$ is not larger than $2n\tau$, and hence the degree of $\text{adj}(E(\theta))$ is not larger than

$$\left(\frac{1}{2}(n-1)n - 1\right) 2n\tau = \mu_1. \tag{2.66}$$

Let us choose $Q(\theta) = \zeta(\theta)^{-2} I_{n-1}$ and define $P(\theta)$ as $(-1)^a \det(E(\theta))P(\theta)$ where a is 0 if $\det(E(\theta)) > 0$ for all $\theta \in \Omega$ or 1 otherwise. One can obtain that $P(\theta)$ is a matrix polynomial of degree not greater than μ_1 satisfying the Lyapunov equation

$$\zeta(\theta)^2 P(\theta) - K_1(\theta)^T P(\theta) K_1(\theta) = \det(E(\theta)) I_{n-1} \tag{2.67}$$

and, hence, (2.56). □

Remark 2.2 *Theorem 2.4 supplies a necessary and sufficient condition for robust consensus of* (2.39) *via finding a Lyapunov function polynomially dependent on the uncertainty. As a requirement for achieving necessity, the degree μ_1 hinges both on the degree of $G(\theta)$ and on the number of agents n.*

The condition of Theorem 2.4 can be established via a convex optimization. Specifically, let $H_i(\theta)$ and $J_i(\theta)$, $i = 1, \ldots, h$, be some auxiliary symmetric matrix

polynomials with size $(n-1)\times(n-1)$, and

$$\begin{aligned} R(\theta) &= P(\theta) - \sum_{i=1}^{h} H_i(\theta)s_i(\theta) \\ T(\theta) &= \zeta(\theta)^2 P(\theta) - K_1(\theta)^T P(\theta)K_1(\theta) - \sum_{i=1}^{h} J_i(\theta)s_i(\theta). \end{aligned} \tag{2.68}$$

The following result provides a sufficient condition for investigating whether (2.39) obtains robust consensus based on conditions of LMIs.

Corollary 2.1 *The condition* (2.56) *satisfies for some symmetric matrix polynomial* $P(\theta)$ *of degree* d *if* $c^* > 0$ *and* c^* *is the solution of the optimization problem as follows.*

$$\begin{aligned} c^* = \sup_{c,H_i,J_i,P} \; & c \\ \text{s.t.} \; & \begin{cases} H_i(\theta) \text{ is SOS} \\ J_i(\theta) \text{ is SOS} \\ R(\theta) - I_{n-1} \text{ is SOS} \\ T(\theta) - cI_{n-1} \text{ is SOS}. \end{cases} \end{aligned} \tag{2.69}$$

Proof Assume that the constraints in (2.69) hold. It follows that

$$\begin{cases} H_i(\theta) \geq 0 \\ J_i(\theta) \geq 0 \\ R(\theta) - I_{n-1} \geq 0 \\ T(\theta) - cI_{n-1} \geq 0 \end{cases} \tag{2.70}$$

for all $\theta \in \mathbb{R}^a$. Since $s_i(\theta) \geq 0$ and $H_i(\theta) \geq 0$, from the third inequality one can obtain

$$\begin{aligned} 0 &\leq R(\theta) - I_{n-1} \\ &= P(\theta) - \textstyle\sum_{i=1}^{h} H_i(\theta)s_i(\theta) - I_{n-1} \\ &\leq P(\theta) - I_{n-1} \end{aligned} \tag{2.71}$$

in which one has that

$$P(\theta) \geq I_{n-1} \ \ \forall \theta \in \Omega. \tag{2.72}$$

Similarly, from the inequality $T(\theta) - cI_{n-1} \geq 0$ one gets

$$\zeta(\theta)^2 P(\theta) - K_1(\theta)^T P(\theta) K_1(\theta) \geq cI_{n-1} \ \ \forall \theta \in \Omega. \tag{2.73}$$

Therefore, if $c > 0$, it implies that (2.56) is satisfied, and hence this theorem holds. □

Corollary 2.1 provides how the condition of Theorem 2.4 can be established via convex programming by using SOS technique. Specifically, since checking whether a matrix polynomial is SOS can be done through solving problem of LMIs as reported in Subsection 1.3.3, it directly follows that the condition of Corollary 2.1 amounts to solving an LMI feasibility test. We can also remark that the conservatism of the condition of Corollary 2.1 hinges on the degrees of $P(\theta)$ and of the multipliers $H_i(\theta)$ and $J_i(\theta)$.

For network with nonnegative weights, let us represent the following preliminary result, which can directly extend to the case of uncertain MASs the condition given in [77] for the case of MASs without considering uncertainty.

Lemma 2.2 *Assume that $G(\theta)$ is nonnegative. The following three statements are equivalent.*

a) The system (2.39) *obtains robust consensus.*

b) for all $\theta \in \Omega$, $D(\theta)$ has exactly one simple eigenvalue 1 *while all the others satisfy $|\lambda| < 1$.*

c) for all $\theta \in \Omega$, the directed graph $\mathcal{G}(\theta)$ contains a spanning tree.

The following result displays how Lemma 2.2 can be used to get a necessary and sufficient condition for robust first-order consensus with nonnegative network weights via investigating the zeros of a polynomial.

Theorem 2.5 *Suppose that $G(\theta)$ is nonnegative. The system* (2.39) *obtains robust consensus if and only if*

$$q_D(\theta) \neq 0 \ \ \forall \theta \in \Omega \tag{2.74}$$

where

$$q_D(\theta) = \left. \frac{dl_D(\lambda, \theta)}{d\lambda} \right|_{\lambda=1} \tag{2.75}$$

and

$$l_D(\lambda, \theta) = \det(\lambda I_n - D(\theta)). \tag{2.76}$$

Proof Assume that $G(\theta)$ is nonnegative. Based on Lemma 2.2 one gets that (2.39) obtains robust consensus if and only if, for all $\theta \in \Omega$, $D(\theta)$ has exactly one simple eigenvalue 1 while all the other eigenvalues with a magnitude smaller than 1.

Since $D(\theta)$ is a stochastic matrix with positive diagonal entries, it implies that every eigenvalue of $D(\theta)$ not equal to 1 has a magnitude smaller than 1, see e.g. [76]. Thus, it just requires to show that the eigenvalue 1 is simple.

This is equivalent to getting that the characteristic polynomial $l_D(\lambda, \theta)$ of $D(\theta)$ can be displayed as

$$l_D(\lambda, \theta) = (\lambda - 1)\xi(\lambda, \theta) \tag{2.77}$$

where

$$\xi(1, \theta) \neq 0 \ \ \forall \theta \in \Omega. \tag{2.78}$$

This last condition coincides with (2.74) due to

$$\frac{dl_D(\lambda, \theta)}{d\lambda} = \xi(\lambda, \theta) + (\lambda - 1)\frac{d\xi(\lambda, \theta)}{d\lambda}. \tag{2.79}$$

Thus, the proof completes. □

Theorem 2.5 gives a necessary and sufficient condition for robust first-order consensus which can be used in the case of nonnegative network weights. This condition needs to investigate whether the polynomial $q_D(\theta)$ is nonzero over Ω.

The condition of Theorem 2.5 can be checked through convex optimization. Indeed, let us introduce

$$q_K(\theta) = \left.\frac{dl_K(\lambda,\theta)}{d\lambda}\right|_{\lambda=1} \tag{2.80}$$

where

$$l_K(\lambda,\theta) = \det(\lambda I_n - K(\theta)). \tag{2.81}$$

Let $a_i(\theta)$, $i = 1,\ldots,h$, be auxiliary polynomials, and let us define

$$b(\theta) = q_K(\theta_0)q_K(\theta) - \sum_{i=1}^{h} a_i(\theta)s_i(\theta) \tag{2.82}$$

where θ_0 can be chosen arbitrarily in Ω.

Corollary 2.2 *Provided that $G(\theta)$ is nonnegative, the condition* (2.74) *is satisfied if $c^* > 0$, where c^* is the solution of the optimization problem as follows.*

$$\begin{aligned} c^* \;=\; & \sup_{a_i,c}\; c \\ & \textit{s.t.} \begin{cases} a_i(\theta) \textit{ is SOS} \\ b(\theta) - c \textit{ is SOS}. \end{cases} \end{aligned} \tag{2.83}$$

Proof Provided that the constraints in (2.83) are satisfied, it implies that

$$\begin{cases} a_i(\theta) \geq 0 \\ b(\theta) - c \geq 0 \end{cases} \tag{2.84}$$

for all $\theta \in \mathbb{R}^a$. Due to $s_i(\theta) \geq 0$ and $a_i(\theta) \geq 0$, from the second inequality we can obtain

$$\begin{aligned} 0 \;&\leq\; b(\theta) - c \\ &=\; q_K(\theta_0)q_K(\theta) - \textstyle\sum_{i=1}^{h} a_i(\theta)s_i(\theta) - c \\ &\leq\; q_K(\theta_0)q_F(\theta) - c \end{aligned} \tag{2.85}$$

which directly implies that

$$q_K(\theta_0)q_K(\theta) \geq c \ \ \forall \theta \in \Omega. \tag{2.86}$$

If $c > 0$, it follows that $q_K(\theta_0)q_K(\theta)$ is positive over Ω. From the continuity of $q_K(\theta)$ and $\theta_0 \in \Omega$, we have that

$$q_K(\theta) \neq 0 \ \ \forall \theta \in \Omega. \tag{2.87}$$

Observe that

$$q_K(\theta) = \zeta(\theta)^n D(\theta) \tag{2.88}$$

and $\zeta(\theta) \neq 0$ for all $\theta \in \Omega$, we conclude that (2.74) holds. □

2.4.2 Second-order Consensus

From (2.44) we have that

$$\begin{aligned} u_1(k) - u_i(k) &= k_1 \left(-L_{ii}(\theta)(x_1(k) - x_i(k)) + \sum_{j=2}^{n} (G_{ij}(\theta) - G_{1j}(\theta))(x_1(k) - x_j(k)) \right) \\ &\quad + k_2 \left(-L_{ii}(\theta)(\varrho_1(k) - \varrho_i(k)) + \sum_{j=2}^{n} (G_{ij}(\theta) - G_{1j}(\theta))(\varrho_1(k) - \varrho_j(k)) \right). \end{aligned} \tag{2.89}$$

It follows that

$$\begin{cases} x_1(k+1) - x_i(k+1) &= x_1(k) - x_i(k) + \varrho_1(k) - \varrho_i(k) \\ \varrho_1(k+1) - \varrho_i(k+1) &= \varrho_1(k) - \varrho_i(k) - k_1 \left(\sum_{j=2}^{n} (L_{ij}(\theta) - L_{1j}(\theta))(x_1(k) - x_j(k)) \right) \\ &\quad - k_2 \left(\sum_{j=2}^{n} (L_{ij}(\theta) - L_{1j}(\theta))(\varrho_1(k) - \varrho_j(k)) \right). \end{cases} \tag{2.90}$$

Thus, (2.45) can be represented as

$$w(k+1) = \tilde{\Gamma}(\theta) w(k) \tag{2.91}$$

where

$$\begin{aligned} w(k) &= (x_1 - x_2, ..., x_1 - x_n, \varrho_1 - \varrho_2, ..., \varrho_1 - \varrho_n)^T \\ \tilde{\Gamma}(\theta) &= \begin{pmatrix} I_{n-1} & I_{n-1} \\ -k_1\tilde{L}(\theta) & I_{n-1} - k_2\tilde{L}(\theta) \end{pmatrix} \\ \tilde{L}(\theta) &= \begin{pmatrix} L_{22}(\theta) - L_{12}(\theta) & \dots & L_{2n}(\theta) - L_{1n}(\theta) \\ \vdots & \ddots & \vdots \\ L_{n2}(\theta) - L_{12}(\theta) & \dots & L_{nn}(\theta) - L_{1n}(\theta) \end{pmatrix}. \end{aligned} \tag{2.92}$$

The following result directly extends to the case of uncertain MASs the condition given in [78] for the case of MASs without uncertainty.

Lemma 2.3 *The system* (2.43) *achieves robust consensus if and only if*

$$\left|\lambda_i\left(\tilde{\Gamma}(\theta)\right)\right| < 1 \ \ \forall i = 1, \dots, 2n-2 \, \forall \theta \in \Omega \tag{2.93}$$

where $\lambda_i\left(\tilde{\Gamma}(\theta)\right)$ *is the* i*-th eigenvalue of* $\tilde{\Gamma}(\theta)$*.*

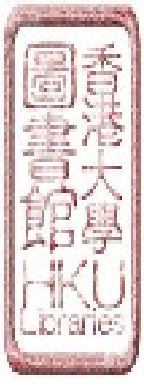
The following result provides a necessary and sufficient condition for checking whether (2.43) achieves robust consensus.

Theorem 2.6 *Let* τ *be the degree of* $G(\theta)$*, and define*

$$\mu_2 = 2n(2n-3)\tau. \tag{2.94}$$

The system (2.43) *obtains robust consensus if and only if there is a symmetric matrix polynomial* $P(\theta) \in \mathbb{R}^{(2n-2)\times(2n-2)}$ *of degree* $d \leq \mu_2$ *such that*

$$\begin{cases} P(\theta) > 0 \\ P(\theta) - \tilde{\Gamma}(\theta)^T P(\theta)\tilde{\Gamma}(\theta) > 0 \end{cases} \quad \forall \theta \in \Omega. \tag{2.95}$$



Proof (Sufficiency) Suppose that (2.95) holds. Due to Lyapunov stability theorem

for discrete-time linear systems, one has that (2.93) holds. Thus, from Lemma 2.3, we can conclude that (2.43) obtains robust consensus.

(Necessity) Suppose that (2.43) obtains robust consensus. From Lemma 2.3 it implies that (2.93) holds, which means that the Lyapunov equation

$$P(\theta) - \tilde{\Gamma}(\theta)^T P(\theta)\tilde{\Gamma}(\theta) = Q(\theta) \tag{2.96}$$

has a unique solution $P(\theta)$ where $P(\theta) > 0$ for all $\theta \in \Omega$ as long as $Q(\theta) > 0$ for all $\theta \in \Omega$. One can rewrite this equation as

$$E(\theta)p(\theta) = q(\theta) \tag{2.97}$$

where $p(\theta)$ and $q(\theta)$ have the $(2n-1)(2n-2)/2$ free entries of $P(\theta)$ and $Q(\theta)$. As the solution $P(\theta)$ exists and is unique, it directly follows that $E(\theta)$ is invertible for all $\theta \in \Omega$, and hence

$$p(\theta) = \frac{\text{adj}(E(\theta))}{\det(E(\theta))} q(\theta). \tag{2.98}$$

Observe the degree of $\tilde{\Gamma}(\theta)$ is not greater than τ, one has that the degree of $E(\theta)$ is not greater than 2τ, and hence the degree of $\text{adj}(E(\theta))$ is not greater than

$$\left(\frac{1}{2}(2n-1)(2n-2) - 1\right) 2\tau = \mu_2. \tag{2.99}$$

Let $Q(\theta) = I_{2n-2}$ and redefine $P(\theta)$ as $(-1)^a \det(E(\theta))P(\theta)$ where a is 0 if $\det(E(\theta)) > 0$ for all $\theta \in \Omega$ or 1 otherwise. One has that $P(\theta)$ is a matrix polynomial of degree not greater than μ_2 that satisfies (2.95). □

Theorem 2.4 gives a necessary and sufficient condition for robust consensus of (2.43) via finding a Lyapunov function polynomially dependent on the uncertainty. This condition can be checked by convex optimization.

Indeed, let $P(\theta)$ be as in Theorem 2.4, and let auxiliary symmetric matrix polynomials $H_i(\theta)$ and $J_i(\theta)$, $i = 1, \ldots, h$, in the size of $(2n-2) \times (2n-2)$, and hence

we can define

$$\begin{aligned} R(\theta) &= P(\theta) - \sum_{i=1}^{h} H_i(\theta) s_i(\theta) \\ T(\theta) &= P(\theta) - \tilde{\Gamma}(\theta)(\theta)^T P(\theta) \tilde{\Gamma}(\theta) - \sum_{i=1}^{h} J_i(\theta) s_i(\theta). \end{aligned} \tag{2.100}$$

The following result supplies a sufficient condition for checking whether (2.43) obtains robust consensus based on LMIs and the proof is analogous to that of Corollary 2.1.

Corollary 2.3 *The condition* (2.95) *holds for some* $P(\theta)$ *of degree* d *if* $c^* > 0$, *where* c^* *is the solution of* (2.69) *with* $R(\theta)$ *and* $T(\theta)$ *which are replaced by those in* (2.100).

2.5 Numerical Examples

2.5.1 Example 1

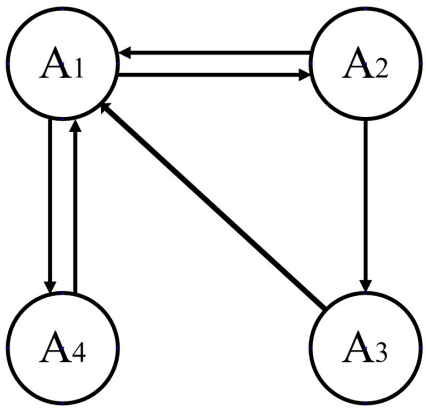


Fig. 2.1: Digraph of a four-agent system

In this example, a four-agent system is considered which is shown in Figure 2.1.

Suppose that the network is affected by an uncertain parameter, specifically,

$$G(\theta) = \begin{pmatrix} 1 & 2-2\theta & 5+\theta & 2+\theta \\ 3\theta & 1 & 0 & 0 \\ 0 & 4-3\theta & 1 & 0 \\ 2+3\theta & 0 & 0 & 1 \end{pmatrix}$$

and Ω is chosen as

$$\Omega = [0, 1].$$

Hence, one has $n = 4$ and $a = 1$. Moreover, Ω can be represented as in (2.2) with

$$s_1(\theta) = \theta(1-\theta).$$

Based on (2.6), the Laplacian matrix $L(\theta)$ is provided by:

$$L(\theta) = \begin{pmatrix} 9 & -2+2\theta & -5-\theta & -2-\theta \\ -3\theta & 3\theta & 0 & 0 \\ 0 & 4-3\theta & 4-3\theta & 0 \\ -2-3\theta & 0 & 0 & 2+3\theta \end{pmatrix}.$$

Observe that all elements of weighted adjacency matrix $G(\theta)$ are positive. Hence, for robust first-order consensus, both Theorem 2.1 and Theorem 2.2 can be exploited in this example. Firstly, let us use Theorem 2.1 by finding a constant matrix function $P_1(\theta)$ satisfying (2.15). By solving (2.22) we can get $c^* = +\infty$, i.e. (2.19) holds with any positive scalar c. Therefore, robust first-order consensus can be obtained.

In order to investigate whether robust first-order consensus can be obtained by this uncertain network with positive weights, we can also use Theorem 2.2. In particular, the polynomial $q(\theta)$ in (2.23) is given by:

$$q(\theta) = 18\theta^3 + 6\theta^2 - 112\theta - 56.$$

Based on condition of Theorem 2.2, robust first-order consensus can be achieved if and only if $q(\theta) \neq 0$ for all $\theta \in [0, 1]$. In this case, it is easy to get that $q(\theta)$ has this property in that $q(\theta)$ is an univariate polynomial whose roots 3.0993, 3.1344 and -6.5670 lie outside $[0, 1]$. Also, let us compute the quantity c^* in (2.29). Set $k = 1$ and let multiplier $g_1(\theta)$ be degree 2, we find $c^* = 56$, hence implying that condition of Theorem 2.2 is satisfied.

Next, let us check whether this uncertain network is able to get robust second-order consensus. We choose

$$\alpha = \beta = 1$$

in the system (2.8), and Theorem 2.3 is used by searching a constant matrix function $P_2(\theta)$ such that (2.33) holds. By computing (2.38) we get $c^* = +\infty$, i.e. (2.35) is satisfied with any positive scalar c. Therefore, robust second-order consensus can be obtained with chosen α and β. In this case, the uncertain extended Laplacian matrix is provided by

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 9 & l_1 & l_2 & l_3 & -9 & l_1 & l_2 & l_3 \\ l_4 & -l_4 & 0 & 0 & l_4 & -l_4 & 0 & 0 \\ 0 & l_5 & -l_5 & 0 & 0 & l_5 & -l_5 & 0 \\ l_6 & 0 & 0 & -l_6 & l_6 & 0 & 0 & -l_6 \end{pmatrix}$$

in which $l_1 = 2 - 2\theta$, $l_2 = 5 + \theta$, $l_3 = 2 + \theta$, $l_4 = 3\theta$, $l_5 = 4 - 3\theta$, $l_6 = 2 + 3\theta$.

2.5.2 Example 2

Considering a network shown in Figure 2.2, an uncertain six-agent system is investigated. In this case, it is assumed that the network is affected by two uncertain

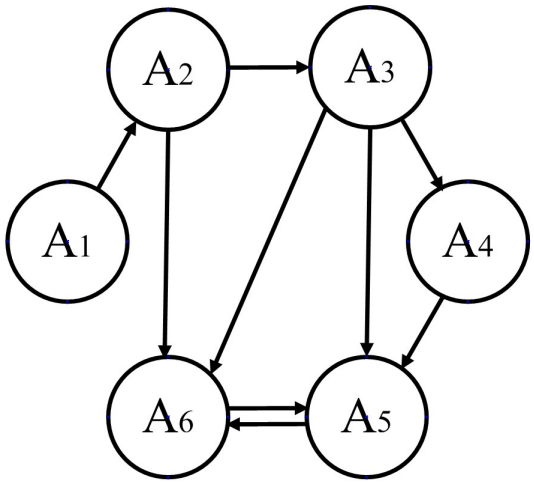


Fig. 2.2: Digraph of a six-agent system

parameters, i.e. θ_1 and θ_2. Indeed, $G(\theta)$ is given by

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 3+2\theta_1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 3-\theta_2 & 1 & 0 & 2\theta_1+\theta_2 & 0 \\ 0 & 0 & 5+2\theta_1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 5 & 1 & 3-4\theta_2 \\ 0 & 5 & 2-3\theta_1 & 0 & 2-\theta_2 & 1 \end{pmatrix}.$$

We choose the set Ω as

$$\Omega = \{\theta : \|\theta\| \leq 1\}.$$

Hence, one has $n = 6$ and $a = 2$. Moreover, Ω can be expressed in (2.2) with

$$s_1(\theta) = 1 - \theta_1^2 - \theta_2^2.$$

To investigate whether this uncertain network is able to achieve robust first-order consensus, observe that it has an either positive or non-positive weighted adjacency matrix, let us employ Theorem 2.1 by searching for a constant matrix function $P_1(\theta)$ under condition (2.15). Via computing (2.22) we can get $c^* = +\infty$, i.e. (2.19) satisfies with any positive scalar c. Therefore, robust first-order consensus can be obtained.

Next, let us establish whether robust second-order consensus can be achieved by this uncertain network. First, we select

$$\alpha = 1,\ \beta = 0.6$$

in the system (2.8), and we exploit Theorem 2.3 by finding a constant matrix function $P_2(\theta)$ satisfying (2.33). By computing (2.38) we get $c^* = -0.1344$, which can not prove (2.35). We repeat this procedure by finding a matrix function $P_2(\theta)$ of degree 2, and for this time we get $c^* = +\infty$, i.e. (2.35) satisfies with any positive scalar c. Therefore, robust second-order consensus can be obtained.

2.5.3 Example 3

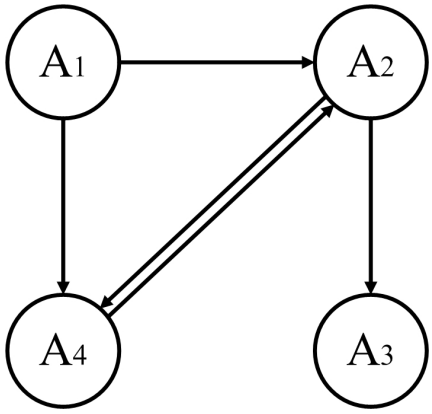


Fig. 2.3: Topology of a four-agent system.

For discrete-time dynamics, let us consider a four-agent system shown in Figure 2.3 with chosen weighted adjacency matrix

$$G(\theta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1+\theta & 1 & 0 & 2 \\ 0 & 1 & 1 & 0 \\ 3+2\theta & 4 & 0 & 1 \end{pmatrix}$$

where θ is constrained in

$$\Omega = [-1, 1].$$

This set can be expressed in (2.2) with

$$s_1(\theta) = 1 - \theta^2.$$

By employing (2.41) we have

$$D(\theta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \dfrac{1+\theta}{4+\theta} & \dfrac{1}{4+\theta} & 0 & \dfrac{2}{4+\theta} \\ 0 & 0.5 & 0.5 & 0 \\ \dfrac{3+2\theta}{8+2\theta} & \dfrac{2}{4+\theta} & 0 & \dfrac{1}{8+2\theta} \end{pmatrix}$$

and, hence,

$$\zeta(\theta) = 8 + 2\theta.$$

First, let us use Corollary 2.1 to investigate whether robust first-order consensus can be obtained. By solving the LMI problem (2.69) with a constant symmetric matrix function $P(\theta)$, one can get $c^* = +\infty$. Hence, from Corollary 2.1 we can conclude that robust first-order consensus can be obtained.

The same conclusion can be achieved using Corollary 2.2 in that $G(\theta)$ is non-negative. Particularly, the polynomial $q_K(\theta)$ in (2.80) is provided by

$$q_K(\theta) = 8\theta^4 + 116\theta^3 + 596\theta^2 + 1248\theta + 832.$$

One can solve the LMI problem (2.83) with a multiplier $a_1(\theta)$ of degree 2, and finds $c^* = 72$. Hence, from Corollary 2.2 one can draw a conclusion that robust first-order consensus can be obtained.

Next, let us exploit Corollary 2.3 to investigate whether robust second-order consensus can be obtained. Specifically, we consider (2.43) with $k_1 = 0.021$ and $k_2 = 0.197$. By computing the LMI problem (2.69) with a symmetric matrix function $P(\theta)$ of degree 1, one can get $c^* = +\infty$. Hence, based on Corollary 2.3, here we conclude that robust second-order consensus can be obtained.

2.5.4 Example 4

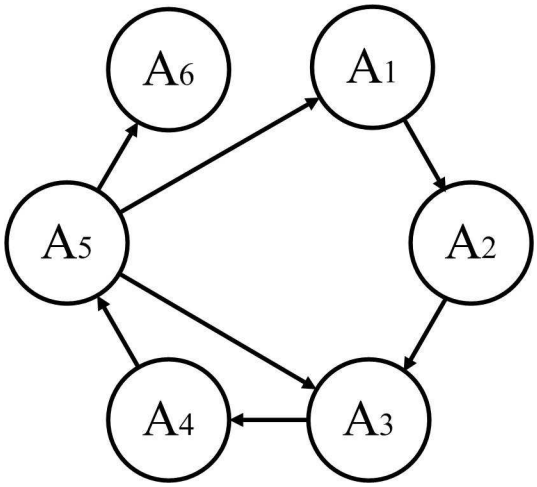


Fig. 2.4: Topology of a six-agent system.

For this example, a six-agent system is considered as shown in Figure 2.4 with

$$G(\theta) = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 \\ 3+\theta_1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 3-\theta_1 & 1 & 0 & \theta_1+\theta_2 & 0 \\ 0 & 0 & 3+\theta_1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1+0.5\theta_2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

where $\theta \in \mathbb{R}^2$ is constrained in

$$\Omega = \left\{\theta \in \mathbb{R}^2 : \ \|\theta\| \leq 1\right\}. \tag{2.101}$$

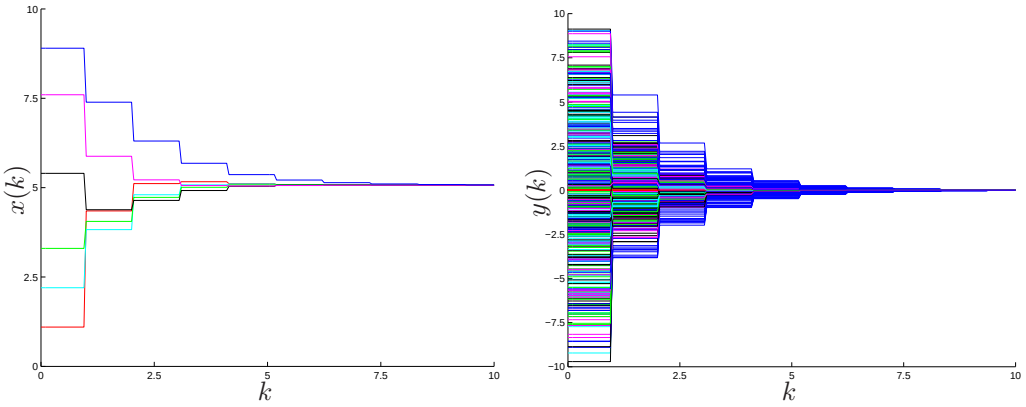
This set can be expressed as in (2.2) with

$$s_1(\theta) = 1 - \theta_1^2 - \theta_2^2.$$

Hence, contrasting with Example 3, this network is affected by two uncertain parameters, and $G(\theta)$ is not nonnegative.

Firstly, Corollary 2.1 is used to establish whether robust first-order consensus

can be obtained. By solving the LMI problem (2.69) with a constant symmetric matrix function $P(\theta)$, one gets $c^* = +\infty$. Hence, based on Corollary 2.1 we can conclude that robust first-order consensus can be obtained. This is also proved by Figure 2.5, which has shown in Figure 2.5(a) a trajectory of $x(k)$ for randomly chosen $\theta \in \Omega$ and $x(0)$, and in Figure 2.5(b) 100 trajectories of $y(k)$, where $y_i(k) = x_i(k) - x_1(k)$, $i = 2, \ldots, 6$, for randomly chosen $\theta \in \Omega$ and $x(0)$.



(a) (b)

Fig. 2.5: Example 4: some trajectories for robust first-order consensus.

Next, let us employ Corollary 2.3 to establish whether robust second-order consensus can be obtained. Particularly, we consider (2.43) by choosing $k_1 = 0.01$ and $k_2 = 0.2$. Then, we solve the LMI problem (2.69) with a symmetric matrix function $P(\theta)$ of degree 1, and get $c^* = +\infty$. Hence, based on Corollary 2.3 we can conclude that robust second-order consensus can be obtained. This is proved by Figure 2.6, which has shown in Figures 2.6a–2.6b a trajectory of $x(k)$ and $\varrho(k)$ for randomly selected $\theta \in \Omega$ and $x(0), \varrho(0)$, and in Figures 2.6(c)–2.6(d) 100 trajectories of $y(k)$ and $z(k)$, where $y_i(k) = x_i(k) - x_1(k)$ and $z_i(k) = \varrho_i(k) - \varrho_1(k)$, $i = 2, \ldots, 6$, for randomly selected $\theta \in \Omega$ and $x(0), \varrho(0)$.

2.6 Summary

In this chapter, robust consensus of MASs with linear dynamics and topological uncertainties is considered, both for continuous-time systems and for discrete-time

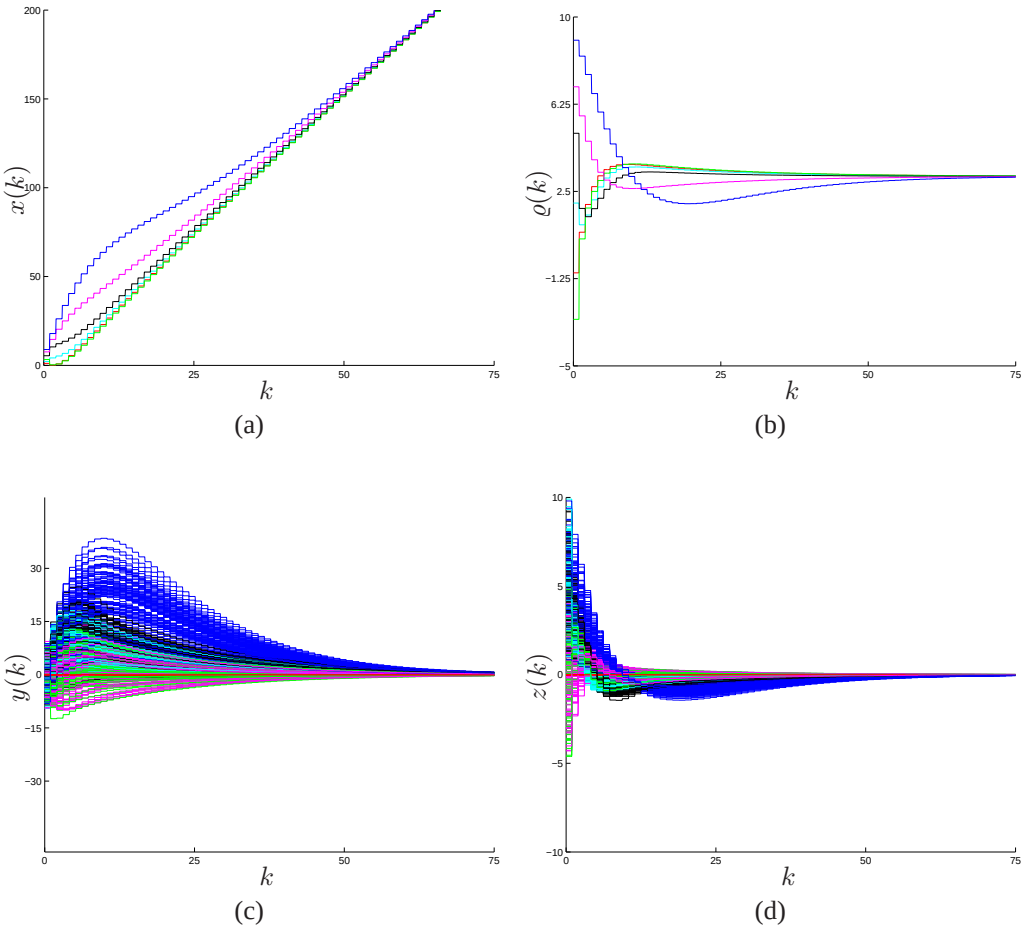


Fig. 2.6: Example 4: some trajectories for robust second-order consensus.

systems. Firstly, necessary and sufficient conditions are provided for robust first-order consensus and for robust second-order consensus in cases of positive and non-positive network weights. In addition, this chapter also considers robust consensus problem with discrete-time dynamics. Necessary and sufficient conditions are also given for robust consensus via searching a polynomial parameter-dependent Lyapunov function. It shows that the necessity can be achieved by computing an upper bound on the degree of candidate Lyapunov function required. Then, a necessary and sufficient condition is given for robust first-order consensus with nonnegative weighted adjacency matrices by checking the zeros of a polynomial. Lastly, by employing SOS technique, these robust consensus conditions can be tested by solving convex optimization problems in terms of LMIs. Four examples are given to demonstrate the usefulness of proposed results.

Chapter 3

Consensus for Nonlinear Dynamics

3.1 Introduction

Firstly, this chapter studies local and global consensus in MASs with nonlinear dynamics. For local consensus, by using HPLFs, a method is provided based on the transformation from the original system into an polytopic system. In addition, for global consensus, another method is given by finding for a suitable PLF. It is shown that this chapter uses more complex Lyapunov function in contrast with the QLFs widely exploited in existing literatures and QLF is demonstrated as a special case of the proposed method.

Furthermore, in this chapter we also consider robust local consensus in MASs with time-varying parametric uncertainties constrained in a polytope. In contrast to existing results with non-convex conditions via exploiting QLF, a novel robust consensus condition is constructed via employing HPLFs, where an uncertain polytopic system is also used to approximate the original system. Furthermore, corresponding solvable conditions in terms of LMIs have been proposed via SMR technique. Finally, polytopic consensus margin problem is introduced and investigated via handling GEVPs. Numerical examples illustrate the effectiveness of the proposed results.

This chapter is organized as follows. Section 3.2 formulates the problems. Section 3.3 gives local and global consensus conditions with nonlinear dynamics. Re-

garding to MASs with time-varying topological uncertainties, Section 3.4 provides robust local consensus conditions for MASs with nonlinear dynamics. It also shows how the polytopic consensus margin can be obtained via handling GEVPs. Section 3.5 demonstrates the proposed results with numeral examples. Section 2.6 summarises this chapter.

3.2 Problem Formulation

In this section, we consider MASs as follows

$$\dot{x}_i(t) = f(x_i(t)) - c\sum_{j=1}^{N} L_{ij}\Gamma x_j(t), \;\; i,j = 1,\ldots,N \tag{3.1}$$

where $x_i \in \mathbb{R}^n$ denotes the state of the i-th agent, N is the number of agents, c denotes the coupling weight, $f(x_i) \in \mathbb{R}^n$ is a nonlinear function, $\Gamma = \text{diag}(\gamma_1,\ldots,\gamma_n) \in \mathbb{R}^{n\times n}$ is a diagonal matrix where $\gamma_i > 0$ means an agent is able to communicate through the i-th state, and L_{ij} stands for the ij-th entry of the Laplacian matrix $L \in \mathbb{R}^{N\times N}$.

The uncertain MASs (3.1) can be rewritten in compact form as

$$\dot{x}(t) = g(x(t)) - c(L\otimes\Gamma)x(t) \tag{3.2}$$

where $x(t) = (x_1(t)^T,\ldots,x_N(t)^T)^T$ and $g(x(t)) = (f(x_1(t))^T,\ldots,f(x_N(t))^T)^T$. Let $s(t) \in \mathbb{R}^n$ be a solution manifold of an isolated node, i.e.

$$\dot{s}(t) = f(s(t)). \tag{3.3}$$

Let us remark that $s(t)$ can exist in various forms like an equilibrium point, a periodic orbit, or a chaotic orbit. Then, two consensus problems are given as follows.

Problem 3.1 *To establish if the MAS* (3.2) *achieves local consensus, i.e. for any* ϵ *there exist* $\kappa(\epsilon)$ *and* $T > 0$ *such that* $\|x_i(0) - x_j(0)\| \le \kappa(\epsilon)$ *implies* $\|x_i(t) -$

$x_j(t)\| \leq \epsilon$ *for all* $t > T$ *and* $i, j = 1, \ldots, N$.

Problem 3.2 *To establish if the MAS* (3.2) *achieves global consensus, i.e. for any* ϵ *there exist* $T > 0$ *such that* $\|x_i(t) - x_j(t)\| \leq \epsilon$ *for all* $t > T$ *and* $i, j = 1, \ldots, N$ *(regardless of* $\|x_i(0) - x_j(0)\|$*).*

3.3 Consensus Conditions with Nonlinear Dynamics

3.3.1 Local Consensus Conditions

For local consensus we introduce the following assumption on $f(x_i)$.

Assumption 3.1 *The function* $f(x_i)$ *is continuously differentiable in the neighbourhood of* $s(t)$.

Remark 3.1 *This assumption is fairly mild since it only requires that the first derivative of the vector field is continuous in a neighbourhood of interested solution manifold.*

By subtracting (3.3) from (3.1), one has the system

$$\dot{y}_i(t) = f(x_i(t)) - f(s(t)) - c\sum_{j=1}^{N} L_{ij}\Gamma y_j(t) \tag{3.4}$$

where $y_i = x_i - s$, $i = 1, \ldots, N$. Then, let us linearize the system (3.4) around $s(t)$ as follows.

$$\dot{y}(t) = (I_N \otimes Df(s(t)))y(t) - c(L \otimes \Gamma)y(t) \tag{3.5}$$

in which $y(t) = (y_1(t)^T, \ldots, y_N(t)^T)^T$ and $Df(s(t)) \in \mathbb{R}^{n\times n}$ is the Jacobian matrix of $f(x_i)$ evaluated for $x_i = s(t)$. Let $z_i = y_1 - y_i$, $i = 2, \ldots, N$, and $z(t) = (z_2(t)^T, \ldots, z_N(t)^T)^T$. We get a reduced system as

$$\begin{aligned} \dot{z}(t) &= A(t)z(t) \\ &= (I_{N-1} \otimes Df(s(t)) - c(\widetilde{L} \otimes \Gamma))z(t) \end{aligned} \tag{3.6}$$

where

$$\widetilde{L} = \begin{pmatrix} L_{22} - L_{12} & \dots & L_{2N} - L_{1N} \\ \vdots & \ddots & \vdots \\ L_{N2} - L_{12} & \dots & L_{NN} - L_{1N} \end{pmatrix}.$$

The definition of local consensus directly derives the following result.

Lemma 3.1 *Suppose that Assumption 3.1 holds. The local consensus of system* (3.2) *can be achieved if the system* (3.6) *is asymptotically stable.*

Proof Based on the definition of y_i one obtains that the local consensus of system (3.2) can be achieved if $|y_i - y_j| \to 0$ whenever the initial condition of y locates near the equilibrium characterized by $y_i^* = y_j^*$ for all i, j. Since $z_i = y_1 - y_i$, the previous condition becomes $|z_i| \to 0$ whenever the initial condition for z locates in a neighbourhood of the origin. This condition can obviously be ensured if the system (3.6) is asymptotically stable in that it is a linear system. □

Next, it will be shown that (3.6) can be transformed into an uncertain polytopic system of following form

$$\begin{cases} \dot{z}(t) &= \widehat{A}(p(t))z(t) \\ p(t) &\in \mathscr{P} \end{cases} \tag{3.7}$$

where $p(t) \in \mathbb{R}^q$ is an uncertain parameter vector, $\mathscr{P}$ is the simplex expressed by

$$\mathscr{P} = \text{co}\{p^{(1)}, \dots, p^{(w)}\}$$

and $\widehat{A}(p(t))$ is described by

$$\widehat{A}(p(t)) = \widehat{A}_0 + \sum_{i=1}^{q} p_i(t)\widehat{A}_i$$

for some $\widehat{A}_0, \widehat{A}_1, \ldots, \widehat{A}_q \in \mathbb{R}^{k\times k}$. This can be done by selecting any bounds $b_{ij}, c_{ij} \in \mathbb{R}$ satisfying

$$b_{ij} \leq A_{ij}(t) \leq c_{ij}\ \forall t \geq 0$$

for all $i, j = 1, \ldots, k$. Let us observe that such bounds exist in that $Df(s(t))$ is continuous. Then, a parameter $p_l(t)$ can be assigned to each entry of $A_{ij}(t)$ by choosing

$$\begin{cases} \widehat{A}_{0,ij} &= b_{ij} \\ \widehat{A}_{l,ij} &= c_{ij} - b_{ij} \end{cases}$$

for the sake of ensuring that the uncertain polytopic system includes (3.6). Obviously, for entries of $A_{ij}(t)$ that are linearly dependent, one can merely introduce one parameter $p_l(t)$.

Robust stability of (3.7) can be checked by HPLFs which is a non-conservative class of Lyapunov functions whose construction can be handled through LMIs, see e.g. [79]. In order to provide a LMI condition based on HPLFs for local consensus of (3.1), the following result is introduced.

Theorem 3.1 *Suppose that Assumption 3.1 holds. The local consensus of* (3.1) *can be achieved if there is a homogeneous function* $v(z)$ *such that*

$$\forall z \neq 0 \quad \begin{cases} 0 < v(z) \\ 0 < -\varrho_i(z)\ \ \forall i = 1, \ldots, w \end{cases} \tag{3.8}$$

where

$$\varrho_i(z) = \dot{v}(z,p)|_{p=p^{(i)}}$$

and

$$\dot{v}(z,p) = \left(\frac{dv(z)}{dz}\right)^T \left(\widehat{A}(p)z\right).$$

Such a $v(z)$ *is a HPLF for* (3.7).

Proof Suppose that (3.8) holds. One can observe that

$$\dot{v}(z,p) = \sum_{i=1}^{w} d_i(p)\varrho_i(z)$$

where $d_1(p), \ldots, d_w(p) \in \mathbb{R}$ satisfy

$$\begin{cases} \sum_{i=1}^{w} d_i(p)p^{(i)} = p \\ \sum_{i=1}^{w} d_i(p) = 1 \\ d_i(p) \geq 0 \ \ \forall i = 1, \ldots, w. \end{cases}$$

Hence, from (3.8), it implies that

$$\dot{v}(z,p) < 0 \ \ \forall z \neq 0$$

i.e. $v(z)$ is a Lyapunov function for (3.7) for all $p \in \mathscr{P}$, in particular a HPLF. Therefore, (3.7) is robustly asymptotically stable, and local consensus of (3.1) can be achieved. □

Let $v(z)$ be a homogeneous polynomial of degree $2m$. One can represent $v(z)$ via the SMR in (1.10) as

$$v(z) = (*)^T V \phi_{\text{hom}}(z,m)$$

where $V \in \mathbb{R}^{l_{\text{hom}}((N-1)n,m) \times l_{\text{hom}}((N-1)n,m)}$ is a symmetric matrix. For the purpose of deriving the LMI condition for local consensus, let us first introduce the following definition.

Definition 3.1 *Let $\widehat{A}^{\#}$ be a matrix such that*

$$\frac{d\phi_{\text{hom}}(z,m)}{dt} = \frac{\partial \phi_{\text{hom}}(z,m)}{\partial z}\widehat{A}z = \widehat{A}^{\#}\phi_{\text{hom}}(z,m). \tag{3.9}$$

Then, $\widehat{A}^{\#}$ is defined to be an extended matrix *of $\widehat{A}$.*

Lemma 3.2 *Let $z^{[m]}$ be the m-th Kronecker power of z, and K_m be the matrix with $z^{[m]} = K_m \phi_{\hom}(z, m)$. Then,*

$$\widehat{A}^{\#} = (K_m^T K_m)^{-1} K_m^T \left(\sum_{i=0}^{m-1} I_{m-1-i} \otimes \widehat{A} \otimes I_i \right) K_m.$$

Let

$$\widetilde{A}_i = \widehat{A}(p^{(i)})$$

and $\widetilde{A}_i^{\#}$ be the extended matrix of $\widetilde{A}_i$. The LMI condition for local consensus is given as follows.

Theorem 3.2 *Suppose that Assumption 3.1 holds. For any $m \geq 1$, let $L(\delta)$ be a linear parametrization of the linear subspace* (1.12) *(see Subsection 1.3.3 for details). The local consensus of* (3.1) *can be achieved if there is a symmetric matrix V and $\delta^{(1)}, \ldots, \delta^{(w)}$ such that*

$$\begin{cases} 0 < V \\ 0 < -\mathrm{he}\left(V \widetilde{A}_i^{\#}\right) - L(\delta^{(i)}) \ \ \forall i = 1, \ldots, w. \end{cases} \tag{3.10}$$

Proof Suppose that (3.10) holds. Via pre- and post-multiplying the first condition in (3.10) by $\phi_{\hom}(z, m)^T$ and $\phi_{\hom}(z, m)$, respectively, one gets that

$$\begin{aligned} 0 &< (*)^T V \phi_{\hom}(z, m) \\ &= v(z) \end{aligned}$$

Hence, it implies that $v(z)$ is positive definite since $\phi_{\hom}(z, m)^T \phi_{\hom}(z, m) > 0$ for all $z \neq 0$. From (3.9), it directly follows that

$$\varrho_i(z) = (*)^T \mathrm{he}(V \widetilde{A}_i^{\#}) \phi_{\hom}(z, m)$$

and from the second LMI condition, one has that $\varrho_i(z)$ is negative definite. Thus, from Theorem 3.1 it follows that $v(z)$ is a HPLF for (3.7), and hence the local consensus of (3.1) can be achieved. □

Let us remark that one can systematically investigate whether there is a symmetric matrix V and $\delta^{(1)}, \ldots, \delta^{(w)}$ such that (3.10) holds. Actually, this is a LMI condition, which amounts to handling with a convex optimization problem (for more details please see [74] and references therein).

3.3.2 Global Consensus Conditions

In order to establish the global consensus of (3.1), (3.4) can be rewritten as

$$\dot{y}(t) = \psi(y(t), s(t)) - c(L \otimes \Gamma)y(t) \tag{3.11}$$

where

$$\begin{cases} y(t) &= (y_1(t)^T, \ldots, y_N(t)^T)^T, \\ \psi(y(t), s(t)) &= (\psi(y_1(t), s(t))^T, \ldots, \psi(y_N(t), s(t))^T)^T \end{cases} \tag{3.12}$$

and

$$\psi(y_i(t), s(t)) = f(y_i(t) + s(t)) - f(s(t)), \;\; i = 1, \ldots, N. \tag{3.13}$$

In this section, we are interested in considering $f(x)$ with following assumption.

Assumption 3.2 *The function $f(x_i)$ is polynomial.*

Remark 3.2 *Sorts of existing results for global consensus like [47,50,80] are under the assumption of QUAD condition (or one-side Lipschits condition). Nevertheless, the QUAD condition is not satisfied for simple nonlinearities like quadratic and cubic polynomial functions. By contrast, Assumption 3.2 includes such nonlinearities, and also includes famous systems such as Lorenz system and Hamiltonian systems.*

In addition, continuous functions can be approximated arbitrarily well by using their polynomial components, which shows that Assumption 3.2 is indeed mild.

The following result is directly from [63].

Lemma 3.3 *Let* $\sigma = (\sigma_1, \ldots, \sigma_N)^T$ *with* $\sigma_i > 0$, $i = 1, \ldots, N$, *and* $\sum_{i=1}^{N} \sigma_i = 1$. *The global consensus of* (3.1) *can be achieved if there is a matrix*

$$M = (I_N - 1_N \sigma^T) \otimes I_n \tag{3.14}$$

such that

$$\lim_{t \to \infty} \|My(t)\| = 0. \tag{3.15}$$

For ease of description, here we consider first the case where $s(t)$ is constant. Following result can be obtained.

Theorem 3.3 *Suppose that Assumption 3.2 holds. The global consensus of* (3.1) *can be achieved if there are* $\varepsilon \in \mathbb{R}$, *a continuously differentiable function* $v(y)$, *and two functions* $u_1(y)$ *and* $u_2(y)$ *such that*

$$\begin{cases} 0 \leq \varphi_i(y) \ \ \forall y \ \ \forall i = 1, \ldots, 4 \\ 0 < \varepsilon \end{cases} \tag{3.16}$$

where

$$\begin{aligned} \varphi_1(y) &= u_1(y) - \varepsilon \\ \varphi_2(y) &= u_2(y) - \varepsilon \\ \varphi_3(y) &= v(y) - u_1(y)\|My\|^2 \\ \varphi_4(y) &= -\dot{v}(y) - u_2(y)\|My\|^2 \end{aligned} \tag{3.17}$$

and

$$\dot{v}(y) = \left(\frac{dv(y)}{dy}\right)^T (\psi(y, s) - c(L \otimes \Gamma)y)\,. \tag{3.18}$$

Proof Suppose that (3.16) holds. From the first condition for $i = 3$ one can get

$$v(y) \geq u_1(y)\|My\|^2$$

and, since $u_1(y)$ is positive from the first condition for $i = 1$,

$$v(y) > 0 \ \ \forall y : \ My \neq 0.$$

In a similar way, for $i = 4$ one can obtain that

$$\dot{v}(y) < 0 \ \ \forall y : \ My \neq 0.$$

Thus, $v(y)$ is positive and its time derivative is negative whenever $My \neq 0$, which implies that (3.15) holds, and therefore global consensus of (3.1) can be achieved. □

Theorem 3.3 gives a condition for global consensus of (3.1) based on finding a Lyapunov function $v(y)$ with (3.15). One can observe that the role of the term My in the definition of $\varphi_3(y)$ and $\varphi_4(y)$ is to require that $v(y)$ and $-\dot{v}(y)$ are positive as long as consensus is not achieved, which implies that $v(y)$ will decrease till My vanishes.

In order to establish the condition of Theorem 3.3 via LMIs, we are also interested in the case where $v(y)$, $u_1(y)$ and $u_2(y)$ are polynomials. Obviously, $v(y)$ has no constant and linear monomials if it has to satisfy (3.16). Thus, let us parametrize $v(y)$, $u_1(y)$ and $u_2(y)$ as follows.

$$\begin{aligned} v(y) &= w_0^T \phi_{\text{pol}}(y, 2m_0) \\ u_i(y) &= w_i^T \phi_{\text{pol}}(y, 2m_i), \ \ i = 1, 2 \end{aligned} \tag{3.19}$$

where, for all $i = 0, 1, 2$, m_i is an integer and w_i is a vector of proper size. Let us represent $\varphi_i(y)$, $i = 1, \ldots, 4$ by the SMR as

$$\varphi_i(y) = (*)^T \left(C_i(\varepsilon, w) + L_i(\delta_i)\right) \phi_{\text{pol}}(y, m_i) \tag{3.20}$$

where $w = (w_0^T, w_1^T, w_2^T)^T$.

Theorem 3.4 *Suppose that Assumption 3.2 holds. The global consensus of* (3.1) *can be achieved if there are ε, w and δ_i, $i = 1, \ldots, 4$, such that*

$$\begin{cases} 0 & \leq & C_i(\varepsilon, w) + L_i(\delta_i) \ \ \forall i = 1, \ldots, 4 \\ 0 & < & \varepsilon. \end{cases} \tag{3.21}$$

Proof Suppose that (3.21) holds. Through pre- and post-multiplying the first condition in (3.21) by $\phi_{\text{pol}}(z, m_i)^T$ and $\phi_{\text{pol}}(z, m_i)$, respectively, one can obtain

$$\begin{aligned} 0 &\leq (*)^T \left(C_i(\varepsilon, w) + L_i(\delta_i)\right) \phi_{\text{pol}}(z, m_i) \\ &= \varphi_i(y) \ \ \forall y \, \forall i = 1, \ldots, 4. \end{aligned}$$

As a result, (3.16) holds, and from Theorem 3.3 one can conclude that the global consensus of (3.1) can be achieved. □

Remark 3.3 *Theorem 3.4 supplies a LMI condition for global consensus of* (3.1). *This condition can be directly extended to the cases where $s(t)$ is a periodic orbit, a chaotic orbit or other bounded solutions by constructing an uncertain polytopic system and by using the LMI condition in Theorem 3.4 at the vertices of the polytope. Hence, the details are omitted here for conciseness.*

3.4 Robust Local Consensus with Time-varying Uncertainties

In this section, robustness of local consensus is investigated for time-varying topological uncertainties. In particular, it is supposed that the weighted adjacency matrix G is affected by topological uncertainty $\theta(t) \in \mathbb{R}^a$, indicating the time-varying disturbs from environment to the system dynamics [45, 81, 82]. And $\theta(t)$ satisfies

$$\theta(t) \in \Omega. \tag{3.22}$$

In this section, we are interested in following class of Ω.

$$\Omega = \mathrm{co}\{\theta^{(1)}, ..., \theta^{(v)}\} \tag{3.23}$$

for some given vectors $\theta^{(1)}, ..., \theta^{(v)} \in \mathbb{R}^a$. Then, let us introduce the uncertain MASs with time-varying uncertainties by

$$\dot{x}_i(t) = f(x_i(t)) - c\sum_{j=1}^{N} L_{ij}(\theta(t))\Gamma x_j(t), \;\; i, j = 1, \ldots, N \tag{3.24}$$

where $x_i \in \mathbb{R}^n$ denotes the state of i-th agent, N denotes the number of agents, c denotes the coupling weight, $f(x_i) \in \mathbb{R}^n$ is a nonlinear function, $\Gamma = \mathrm{diag}(\gamma_1, \ldots, \gamma_n) \in \mathbb{R}^{n\times n}$ denotes a diagonal matrix where $\gamma_i > 0$ stands for the agents communicating through their i-th states. $L_{ij}(\theta(t))$ is the ij-th entry of the uncertain Laplacian matrix $L(\theta(t)) \in \mathbb{R}^{N\times N}$ given by $L_{ij}(\theta(t)) = -G_{ij}(\theta(t))$ for all $i \neq j$ and by $L_{ii}(\theta(t)) = -\sum_{j=1,\ j\neq i}^{N} L_{ij}(\theta(t))$.

Linear perturbation in network of MASs is widely employed in literatures where $G_{ij}(\theta(t))$ is a linear function [45, 64, 83]. Thus, the uncertain Laplacian matrix can be represented as

$$L(\theta(t)) = L_0 + \sum_{i=1}^{a} \theta_i(t) L_i.$$

The uncertain MAS (3.24) can be rewritten in a compact form

$$\dot{x}(t) = g(x(t)) - c(L(\theta(t)) \otimes \Gamma)x(t) \tag{3.25}$$

where $x(t) = (x_1(t)^T, \ldots, x_N(t)^T)^T$ and $g(x(t)) = (f(x_1(t))^T, \ldots, f(x_N(t))^T)^T$. Let $s(t) \in \mathbb{R}^n$ be a solution of an isolated node, i.e.

$$\dot{s}(t) = f(s(t)). \tag{3.26}$$

Let us observe that $s(t)$ could be an equilibrium point, a periodic orbit, or a chaotic oscillator, etc. Now, we can propose the robust local consensus problem as follows.

Problem 3.3 *To establish whether the uncertain MAS* (3.25) *achieves robust local consensus, i.e. for any* ϵ *there exist* $\kappa(\epsilon)$ *and* $T > 0$ *such that* $\|x_i(0) - x_j(0)\| \leq \kappa(\epsilon)$ *implies* $\|x_i(t) - x_j(t)\| \leq \epsilon$ *for all* $\theta(t) \in \Omega$, $t > T$ *and* $i, j = 1, \ldots, N$.

Another related problem of great academic interests is the consensus margin problem, which will be investigated in Subsection 3.4.3.

3.4.1 System Approximation

First, we note that $f(x_i)$ satisfies the Assumption 3.1. Let $\theta(t) \in \Omega$ be defined by (3.23).

Remark 3.4 *The uncertain parameter* $\theta(t)$ *denotes a polytopic uncertainty which is a representative form both for time-varying system and for time-invariant system in the area of robust control [83–85].*

Let us observe that $\sum_{j=1}^{N} L_{ij}(\theta(t))\Gamma s(t) = 0$. By subtracting (3.26) from (3.24), one can get

$$\dot{y}_i(t) = f(x_i(t)) - f(s(t)) - c\sum_{j=1}^{N} L_{ij}(\theta(t))\Gamma y_j(t) \tag{3.27}$$

where $y_i = x_i - s$, $i = 1, \ldots, N$. For local consensus, the dynamics of the system can be used locally about $s(t)$ in the case without uncertainty [47, 86, 87]. For the uncertain system (3.27), it can also be displayed as

$$\dot{y}(t) = (I_N \otimes Df(s(t)))y(t) - c(L(\theta(t)) \otimes \Gamma)y(t) \tag{3.28}$$

where $y(t) = (y_1(t)^T, \ldots, y_N(t)^T)^T$ and $Df(s(t)) \in \mathbb{R}^{n\times n}$ is the Jacobian matrix of $f(x_i)$ evaluated for $x_i = s(t)$. Observe $\mathbf{1}_N$ is the right eigenvector of $L(\theta(t))$ with respect to eigenvalue zero, let $\eta^T = (\eta_1, ..., \eta_N) \in \mathbb{R}^{1\times N}$ be the left eigenvector of $L(\theta(t))$ corresponding to eigenvalue zero, and $\sum_{i=1}^{N} \eta_i = 1$. A disagreement variable can be defined as follows:

$$z(t) = y(t) - ((\mathbf{1}_N \eta^T) \otimes I_n)y(t) \tag{3.29}$$

where $z(t) \in \mathbb{R}^{nN}$ has a property that $(\eta^T \otimes I_n)z(t) = \mathbf{0}_n$. Define

$$M = (I_N - \mathbf{1}_N \eta^T) \otimes I_n. \tag{3.30}$$

One can observe that matrix M commutes with matrices $I_N \otimes Df(s(t))$ and $c(L(\theta(t)) \otimes \Gamma)$, then an uncertain disagreement system can be constructed as follows:

$$\dot{z}(t) = (I_N \otimes Df(s(t)) - cL(\theta(t)) \otimes \Gamma)z(t). \tag{3.31}$$

Lemma 3.4 *Suppose that Assumption 3.1 holds. The robust local consensus of system* (3.28) *can be achieved if and only if system* (3.31) *is asymptotically stable.*

Proof (Necessity) It is obvious that the robust local consensus of system (3.28) can be achieved if $|y_i - y_j| \to \mathbf{0}_n$ whenever the initial condition for y locates in a neighbourhood of the equilibrium where $y_i^* = y_j^*$ for all i, j. Assume $\lim_{t\to\infty} y(t) \to$

$(\tau(t)^T, ..., \tau(t)^T)^T = \mathbf{1}_N \otimes \tau(t)$. One has

$$\begin{aligned}\lim_{t\to\infty} z(t) &= ((I_N - \mathbf{1}_N\eta^T) \otimes I_n) \times (\mathbf{1}_N \otimes \tau(t)) \\ &= ((I_N - \mathbf{1}_N\eta^T)\mathbf{1}_N) \otimes \tau(t) = \mathbf{0}_{nN}.\end{aligned}$$

(Sufficiency) One can observe that there exist matrices $\Upsilon \in \mathbb{R}^{N\times(N-1)}$ and $\Psi \in \mathbb{R}^{(N-1)\times N}$ such that

$$\begin{pmatrix} \eta^T \\ \Psi \end{pmatrix} L(\theta(t))(\mathbf{1}_N \ \ \Upsilon) = \begin{pmatrix} 0 & \mathbf{0}_{N-1}^T \\ \mathbf{0}_{N-1} & \Xi(\theta(t)) \end{pmatrix}$$

where $\Xi \in \mathbb{R}^{(N-1)\times(N-1)}$ is a matrix function in $\theta(t)$. For system (3.28), pre-multiplying by $\begin{pmatrix} \eta^T \\ \Psi \end{pmatrix} \otimes I_n$, the first n rows generate that

$$\dot{\xi} = Df(s(t))\xi(t) \tag{3.32}$$

where $\xi(t) \in \mathbb{R}^n$. Suppose system (3.31) is asymptotically stable, it is obvious that

$$y(t) \to (\xi(t)^T, \xi(t)^T, ..., \xi(t)^T)^T = \mathbf{1}_N \otimes \xi(t).$$

This completes this proof. □

Lemma 3.5 *Suppose that Assumption 3.1 holds. The robust local consensus of system* (3.25) *can be achieved if asymptotic stability is ensured for following polytopic system.*

$$\begin{cases} \dot{z}(t) &= \widehat{A}(p(t))z(t) \\ p(t) &\in \mathscr{P} \end{cases} \tag{3.33}$$

where $p(t) \in \mathbb{R}^q$ stands for an uncertain parameter vector, $\mathscr{P}$ is the polytope defined by

$$\mathscr{P} = \mathrm{co}\{p^{(1)}, \ldots, p^{(\upsilon)}\}$$

and $\widehat{A}(p(t))$ *satisfies*

$$\widehat{A}(p(t)) = \widehat{A}_0 + \sum_{i=1}^{q} p_i(t)\widehat{A}_i$$

and $\widehat{A}_0, \widehat{A}_1, \ldots, \widehat{A}_q \in \mathbb{R}^{q\times q}$.

Proof Let us define

$$D(t) = I_N \otimes Df(s(t)).$$

Any suitable bounds $b_{ij}, c_{ij} \in \mathbb{R}$ can be properly chosen such that

$$b_{ij} \leq D_{ij}(t) \leq c_{ij}\ \forall t \geq 0$$

for all $i, j = 1, \ldots, k$ and $k = nN$. Obviously, such bounds always exist in that $Df(s(t))$ is continuous. Then, let us define $\iota(t) \in \mathbb{R}^b$ such that

$$\iota \in \mathscr{I} = \mathrm{co}\{\iota^{(1)}, ..., \iota^{(c)}\}$$

and for each entry of $D_{ij}(t)$, a parameter $\iota_l(t)$ is defined by selecting

$$\begin{cases} \widehat{D}_{0,ij} &= b_{ij} \\ \widehat{D}_{l,ij} &= c_{ij} - b_{ij} \end{cases}$$

such that $D(t)$ is entirely contained in the uncertain polytopic system. It is clear that for entries of $D_{ij}(t)$ that are linearly dependent, merely one parameter $\iota_l(t)$ is required. Then, system (3.31) can be represented as

$$\dot{z}(t) = A\left(\sum_{1=1}^{b} D_i\iota_i(t), \sum_{i=1}^{a} L_i\theta_i(t)\right)z(t) \tag{3.34}$$

where function A is linear on $\iota_i(t)$, for all $i = 1, ..., b$ and also linear on $\theta_i(t)$, for all $i = 1, ..., a$. One can get a new time-varying variable $\hat{p}(t) \in \mathbb{R}^{a+b}$ constrained in $\widehat{\mathscr{P}} = \mathrm{co}\{\hat{p}^{(1)}, \ldots, \hat{p}^{(\hat{v})}\}$ such that system (3.34) can be further equivalently displayed as

$$\dot{z}(t) = \widehat{A}(\hat{p}(t))z(t).$$

which completes the proof. □

Remark 3.5 *In an overwhelming number of existing literatures, local consensus conditions are provided based on the solution manifold $s(t)$, thus leaving it a non-convex consensus condition which is rarely tractable. This lemma provides an essential transformation which points out a useful way to make conditions of robust local consensus solvable by convex approaches given by Section 1.3.3. Nevertheless, we have to admit that conservatism arises from the gap between the polytope $\mathscr{I}$ and the manifold $s(t)$. Approaches without utilizing this approximation will also be discussed in Subsection 3.4.2.*

Based on Lemma 3.5, robust local consensus problem changes to a robust stability problem of (3.33), which can be properly established by a non-conservative Lyapunov stability approach, i.e., using HPLFs. Moreover, robust local consensus conditions can be examined by handling a LMI feasibility test.

3.4.2 Conditions via Using HPLF

Let us review the definition of HPLF provided in Theorem 3.1, and let $v(z)$ be a HPLF of degree $2m$ for the system (3.33). Then, following theorem supplies a robust local consensus condition for system (3.33).

Theorem 3.5 *Under Assumption 1, if there is a continuously differentiable homogeneous function $v(z)$ fulfilling*

$$\begin{cases} 0 < v(z) \\ 0 < -\mu_i(z) \ \ \forall i = 1, \ldots, v, \end{cases} \qquad \forall z \neq 0 \tag{3.35}$$

where

$$\mu_i(z) = \dot{v}(z,p)|_{p=p^{(i)}}$$

and

$$\dot{v}(z,p) = \left(\frac{dv(z)}{dz}\right)^T \left(\widehat{A}(p)z\right).$$

Then, function $v(z)$ is a HPLF for (3.33) *and the robust local consensus of* (3.24) *can be achieved.*

Proof Since $p(t) \in \mathbb{R}^q$ and $\mathscr{P}$ is a polytope described by $\mathscr{P} = \mathrm{co}\{p^{(1)}, \ldots, p^{(\upsilon)}\}$, one can get $d_1(p), \ldots, d_w(p) \in \mathbb{R}$ such that

$$\widehat{A}(p(t)) = \sum_{i=1}^{w} d_i(p)\widehat{A}(p^{(i)})$$

where $d_1(p), \ldots, d_w(p) \in \mathbb{R}$ satisfy

$$\begin{cases} \sum_{i=1}^{w} d_i(p)p^{(i)} = p \\ d_i(p) \geq 0 \ \ \forall i = 1, \ldots, \upsilon \\ \sum_{i=1}^{w} d_i(p) = 1. \end{cases}$$

Suppose that (3.35) holds. One has that

$$\begin{aligned} \dot{v}(z,p) &= \left(\frac{dv(z)}{dz}\right)^T \left(\sum_{i=1}^{w} d_i(p)\widehat{A}(p^{(i)})z\right) \\ &= \sum_{i=1}^{w} d_i(p) \left(\frac{dv(z)}{dz}\right)^T \left(\widehat{A}(p^{(i)})z\right) \\ &= \sum_{i=1}^{w} d_i(p)\mu_i(z) \end{aligned}$$

which yields that

$$\dot{v}(z,p) < 0 \ \ \forall z \neq 0$$

Hence, for all $p \in \mathscr{P}$, $v(z)$ is a HPLF for (3.33). Therefore, (3.33) is asymptotically robustly stable, and robust local consensus of (3.24) can be achieved. □

Remark 3.6 *For Theorem 3.5, it is worth noting that*

- *Theorem 3.5 gives conditions for robust local consensus, and it avoids calculating the eigenvalues of Laplacian matrix which is needed in the literatures.*

In addition, HPLF is used and provides a less conservative condition than QLFs widely adopted by literatures, thus proposing a possible way to investigate topological conditions by graph theory.

- *For nonlinear time-varying uncertainties, the approach of HPLF can hardly be used. Nevertheless, provided that $G(\theta)$ is polynomial function of θ, sufficient conditions can be given by exploiting polynomial parameter-dependent Homogeneous Lyapunov function (PPD-HLF), i.e., constructing a Lyapunov function which has a polynomial dependence on uncertain parameter θ.*

- *Sufficient conditions can also be provided by PPD-HLF in the case where the transformation is not used in Lemma 3.5. But this approach can hardly provide solvable conditions such as LMI conditions since $s(t)$ is involved. Moreover, for the sake of proposing some solvable conditions, various assumptions are required while the conservatism could be elevated, such as assuming $|s(t)|_\infty < c$, where c is a positive constant.*

One intuitive yet effective way for checking whether a homogeneous polynomial is nonnegative consists of examining whether it is a SOS polynomial, which can be equivalently represented as a LMI feasibility test (Refer to Subsection 1.3.3).

According to Theorem 3.1, we can display the HPLF $v(z)$ via SMR in (1.10) as follows

$$v(z) = (*)^T V \phi_{\text{hom}}(z, m) \tag{3.36}$$

where $V \in \mathbb{R}^{l_{\text{hom}}((N-1)n,m)\times l_{\text{hom}}((N-1)n,m)}$ is a symmetric matrix. Based on Definition 3.1 and Lemma 3.2, for matrix for $\widehat{A}$, one can get the corresponding extended matrix $\widehat{A}^{\#}$ which is given by

$$\widehat{A}^{\#} = (K_m^T K_m)^{-1} K_m^T \left(\sum_{i=0}^{m-1} I_{m-1-i} \otimes \widehat{A} \otimes I_i \right) K_m. \tag{3.37}$$

Let us denote that

$$\widetilde{A}_i = \widehat{A}\left(p^{(i)}\right)$$

and let $\widetilde{A}_i^{\#}$ be the extended matrix of $\widetilde{A}_i$. LMI conditions for robust local consensus can be proposed as follows.

Theorem 3.6 *Under Assumption 3.1, the robust local consensus of* (3.24) *can be achieved if there exist a symmetric matrix* V *and* $\delta^{(1)}, \ldots, \delta^{(\upsilon)}$ *such that*

$$\begin{cases} 0 < V \\ 0 > F(V,\ \delta^{(i)})\ \ \forall i = 1, \ldots, \upsilon. \end{cases} \tag{3.38}$$

where

$$F(V,\ \delta^{(i)}) = \mathrm{he}\left(V\widetilde{A}_i^{\#}\right) + L\left(\delta^{(i)}\right).$$

Proof Provided that (3.38) holds, via pre- and post-multiplying the first condition in (3.38) by $\phi_{\text{hom}}(z,m)^T$ and $\phi_{\text{hom}}(z,m)$, respectively, one can get that

$$\begin{aligned} 0 &< (*)^T V \phi_{\text{hom}}(z,m) \\ &= v(z) \end{aligned}$$

which directly yields that $v(z)$ is positive definite since the square of power vector $\phi_{\text{hom}}(z,m)^T\phi_{\text{hom}}(z,m) > 0$ for all $z \neq 0$. Moreover, from (3.9) one has that

$$\mu_i(z) = (*)^T\left(V\widetilde{A}_i^{\#} + \left(V\widetilde{A}_i^{\#}\right)^T\right)\phi_{\text{hom}}(z,m)$$

and based on the second LMI condition, one can obtain that

$$\mu_i(z) < 0.$$

Thus, by condition (3.38), $v(z)$ is verified to be a HPLF for (3.33). Therefore, from Theorem 3.5, the robust local consensus of (3.24) can be achieved which ends the proof. □

Remark 3.7 *One can systematically check whether there exist a symmetric matrix* V *and* $\delta^{(1)}, \ldots, \delta^{(\upsilon)}$ *such that* (3.38) *holds. Actually, this is a LMI condition, which amounts to tackling with a convex optimization problem.*

3.4.3 Polytopic Consensus Margin

Subsection 3.4.2 has given the answer how the robust local consensus with polytopic uncertainties can be achieved. Another question arises reasonably that what is the largest level of polytopic uncertainties such that the robustness of local consensus maintains. In order to cope with this problem, let us first bring in following definitions.

Definition 3.2 ζ_{2m}^{P} *is called* $2m$-HPLF polytopic consensus margin *for system* (3.24) *if there is a HPLF* v *with degree* $2m$ *for system* (3.24) *such that*

$$\zeta_{2m}^{P} = \sup\Big\{\zeta \in \mathbb{R} : \theta(t) \in \mathrm{co}\Big\{\zeta\theta^{(1)}, ..., \zeta\theta^{(\upsilon)}\Big\}\Big\}.$$

Of special worth is another denotation which comes from a special instance of above definition, considering the polytope Ω as the unit ℓ_∞ box.

Definition 3.3 ζ_{2m}^{∞} *is called* $2m$-HPLF ℓ_∞ consensus margin *for system* (3.24) *if there exists a HPLF* v *with degree* $2m$ *for* (3.24) *such that*

$$\zeta_{2m}^{\infty} = \sup\Big\{\zeta \in \mathbb{R} : \|\theta(t)\|_\infty \leq \zeta\Big\}.$$

Let us propose the problem of estimating ζ_{2m}^{∞} as follows.

Problem 3.4 *(*$2m$*-HPLF* ℓ_∞ *consensus margin problem) To search for the lower bound of* ζ_{2m}^{∞} *if there exists a HPLF* v *with degree* $2m$ *for* (3.24).

System (3.33) can be rewritten with $\theta(t) = p(t) \in \mathbb{R}^a$ and $\Omega = \mathscr{P}$ as follows.

$$\begin{cases} \dot{z}(t) &= \widehat{A}(\theta(t))z(t) \\ \theta(t) &\in \Omega. \end{cases} \tag{3.39}$$

Denote the vertices of the unit ℓ_∞ ball by $\nu^{(1)}, ..., \nu^{(2^a)}$, and let

$$\bar{A}_i = \widehat{A}(\theta^{(i)}) - \widehat{A}_0, \;\; i = 1, ..., 2^a,$$

and $\bar{A}_i^{\#}$, $i = 1, ..., 2^a$, be the corresponding extended matrix of $\bar{A}_i$ (see Definition 3.1). Following result gives a desirable way which consists of a quasi-convex optimization to compute the $2m$-HPLF ℓ_∞ consensus margin.

Theorem 3.7 *Define*

$$\hat{\zeta}_{2m}^{\infty} = \frac{1}{\vartheta^*} \tag{3.40}$$

where integer $m \geq 1$, ϑ^* can be gained from

$$\vartheta^* = \inf_{\vartheta,\, V,\, \delta^{(0)},\ldots,\delta^{(2^a)}} \vartheta$$

$$s.t. \begin{cases} 0 < \vartheta \\ 0 < V \\ 0 < -\mathrm{he}\Big(V\widehat{A}_0^{\#}\Big) - L\Big(\delta^{(0)}\Big) \\ 0 < \vartheta\Big(-\mathrm{he}\Big(V\widehat{A}_0^{\#}\Big) - L\Big(\delta^{(0)}\Big)\Big) \\ \qquad -\mathrm{he}\Big(V\bar{A}_i^{\#}\Big) - L\Big(\delta^{(i)}\Big) \;\forall i = 1, \ldots, 2^a \end{cases} \tag{3.41}$$

and $L(\cdot)$ is a linear parametrization of $\mathcal{L}$ in (1.12)*. Then $\hat{\zeta}_{2m}^{\infty}$ is the lower bound of ζ_{2m}^{∞}, i.e. $\hat{\zeta}_{2m}^{\infty} \leq \zeta_{2m}^{\infty}$.*

Proof Suppose that (3.41) holds. By pre- and post-multiplying the second LMI condition in (3.41) by $\phi_{\mathrm{hom}}(z, m)^T$ and $\phi_{\mathrm{hom}}(z, m)$, respectively, one gets that

$$0 < (*)^T V \phi_{\mathrm{hom}}(z, m)^T$$

which implies $v(z)$ is positive definite since $\phi_{\hom}(z,m)^T\phi_{\hom}(z,m) > 0$ for all $z \neq 0$. In addition, the derivative of $v(z)$ for $\theta = \vartheta^{-1}\nu^{(i)}$ can be obtained by

$$\begin{aligned}\dot{v}(z)|_{\theta=\vartheta^{-1}\nu^{(i)}} &= (*)^T\mathrm{he}\Big(V\Big(\widehat{A}_0^{\#} + \vartheta^{-1}\bar{A}_i^{\#}\Big)\Big)\phi_{\hom}(z,m)\\ &= \vartheta^{-1}(*)^T\Big(\vartheta\mathrm{he}\Big(V\widehat{A}_0^{\#}\Big) + \mathrm{he}\Big(V\bar{A}_i^{\#}\Big)\Big)\phi_{\hom}(z,m)\\ &= \vartheta^{-1}(*)^T\Big(\vartheta\Big(\mathrm{he}\Big(V\widehat{A}_0^{\#}\Big) + L\Big(\delta^{(0)}\Big)\Big)\\ &\quad +\mathrm{he}\Big(V\bar{A}_i^{\#}\Big) + L\Big(\delta^{(i)}\Big)\Big)\phi_{\hom}(z,m).\end{aligned} \tag{3.42}$$

Thus, from the last constraint in (3.41) one can obtain

$$\dot{v}(z)|_{\theta=\vartheta^{-1}\nu^{(i)}} < 0 \ \ \forall i = 1, ..., 2^a.$$

Based on this, one can also obtain that $\dot{v}(z)$ is negative definite for all $\theta(t)$ in set

$$\Big\{\theta(t) \in \mathbb{R}^a : \|\theta(t)\|_\infty \leq \vartheta^{-1}\Big\}.$$

Therefore, one has $\hat{\zeta}_{2m}^{\infty} \leq \zeta_{2m}^{\infty}$. This proof is thus completed. □

Remark 3.8 *Theorem 3.7 gives a lower bound for $2m$-HPLF ℓ_∞ consensus margin ζ_{2m}^{∞}. In specific, one gets $\hat{\zeta}_{2m}^{\infty} = \zeta_{2m}^{\infty}$ when $(nN,\ 2m)$ is in certain sets, e.g. $\{(nN, 2) :\ nN \in \mathbb{N}\}$, $\{(2, 2m) :\ m \in \mathbb{N}\}$ and $\{(3, 4)\}$ [88]. These sets are associated with the Hilberts 17th problem concerning on the gap between SOS polynomials and positive polynomials.*

A simple result can be given directly from Theorem 3.7 when we consider $a = 1$ and $\theta \in [0, \psi]$. Analogous with ζ_{2m}^{∞}, we define ψ_{2m}^{∞} for the case of system (3.33) with scalar uncertainty broadly adopted in literatures.

Corollary 3.1 *Let us define*

$$\hat{\psi}_{2m}^{\infty} = \frac{1}{\vartheta^*} \tag{3.43}$$

where integer $m \geq 1$, ϑ^* is the solution of

$$\vartheta^* = \inf_{\vartheta,\ V,\ \delta^{(1)},\delta^{(2)}} \vartheta$$
$$s.t. \begin{cases} 0 < V \\ 0 < -\mathrm{he}\Big(V\widehat{A}_0^{\#}\Big) - L\Big(\delta^{(1)}\Big) \\ 0 < \vartheta\Big(-\mathrm{he}\Big(V\widehat{A}_0^{\#}\Big) - L\Big(\delta^{(1)}\Big)\Big) \\ \qquad -\mathrm{he}\Big(V\bar{A}_1^{\#}\Big) - L\Big(\delta^{(2)}\Big) \end{cases} \tag{3.44}$$

and $L(\cdot)$ is a linear parametrization of $\mathcal{L}$ in (1.12)*. Then $\hat{\psi}_{2m}^{\infty}$ is the lower bound of ψ_{2m}^{∞}, i.e. $\hat{\psi}_{2m}^{\infty} \leq \psi_{2m}^{\infty}$.*

3.5 Numerical Examples

In this section, some examples are provided to illustrate the proposed methods.

3.5.1 Example 1

Let us consider a coupled system as an example for local consensus where each agent evolves in a second order dynamics. The nonlinear function $f(x)$ in model (3.1) is given by

$$f(x_i) = \begin{pmatrix} x_{i1} - x_{i2} - x_{i1}(x_{i1}^2 + x_{i2}^2) \\ x_{i1} + x_{i2} - x_{i2}(x_{i1}^2 + x_{i2}^2) \end{pmatrix}$$

where $x_i = (x_{i1}, x_{i2})^T$, $i = 1, 2$. The linear part of (3.1) is described by the constants

$$c = 1,\ \ \Gamma = I_2,\ \ G = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}.$$

One has that (3.3) holds with

$$s(t) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \ \text{ or } \ s(t) = \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}.$$

Let us consider the problem of establishing local consensus for the second solution of $s(t)$, i.e. for the periodic orbit. The matrix $A(t)$ in (3.6) is given by

$$A(t) = \begin{pmatrix} -2 - 3\cos^2 t - \sin^2 t & -1 - 2\cos t \sin t \\ 1 - 2\cos t \sin t & -2 - \cos^2 t - 3\sin^2 t \end{pmatrix}.$$

As described in Subsection 3.3.1, one can build (3.6) in an uncertain polytopic system. Indeed, by selecting $p_1 = \cos^2 t$ and $p_2 = \cos t \sin t$, it yields that $\widehat{A}(p)$ in (3.7) is given by

$$\widehat{A}(p) = \begin{pmatrix} -3 - 2p_1 & -1 - 2p_2 \\ 1 - 2p_2 & -5 + 2p_1 \end{pmatrix}.$$

Observe that $p_1 \in [0, 1]$, $p_2 \in [-0.5, 0.5]$. Thus, the polytope $\mathscr{P}$ can be expressed by

$$\mathscr{P} = \text{co}\left\{ \begin{pmatrix} 1 \\ 0.5 \end{pmatrix}, \begin{pmatrix} 1 \\ -0.5 \end{pmatrix}, \begin{pmatrix} 0 \\ 0.5 \end{pmatrix}, \begin{pmatrix} 0 \\ -0.5 \end{pmatrix} \right\}.$$

One can get that the LMI condition (3.10) holds and hence local consensus can be achieved according to Theorem 3.2. In particular, a HPLF in this case can be found easily by $v(z) = z_{21}^4 + z_{21}^2 z_{22}^2 + z_{22}^4$.

Figure 3.1 shows some simulations for this case. In particular, the first subfigure displays the trajectory of $x(t)$ initializing from $x(0) = (1, 2, -1, -2)^T$, while the second subfigure exhibits 100 trajectories for $z(t)$ with initial conditions randomly chosen in $[-10, 10]^4$.

3.5.2 Example 2

Let us consider (3.1) as an example for global consensus with

$$f(x_i) = \begin{pmatrix} -x_{i2} \\ -x_{i1} - x_{i1}^3 - x_{i2} \end{pmatrix}$$

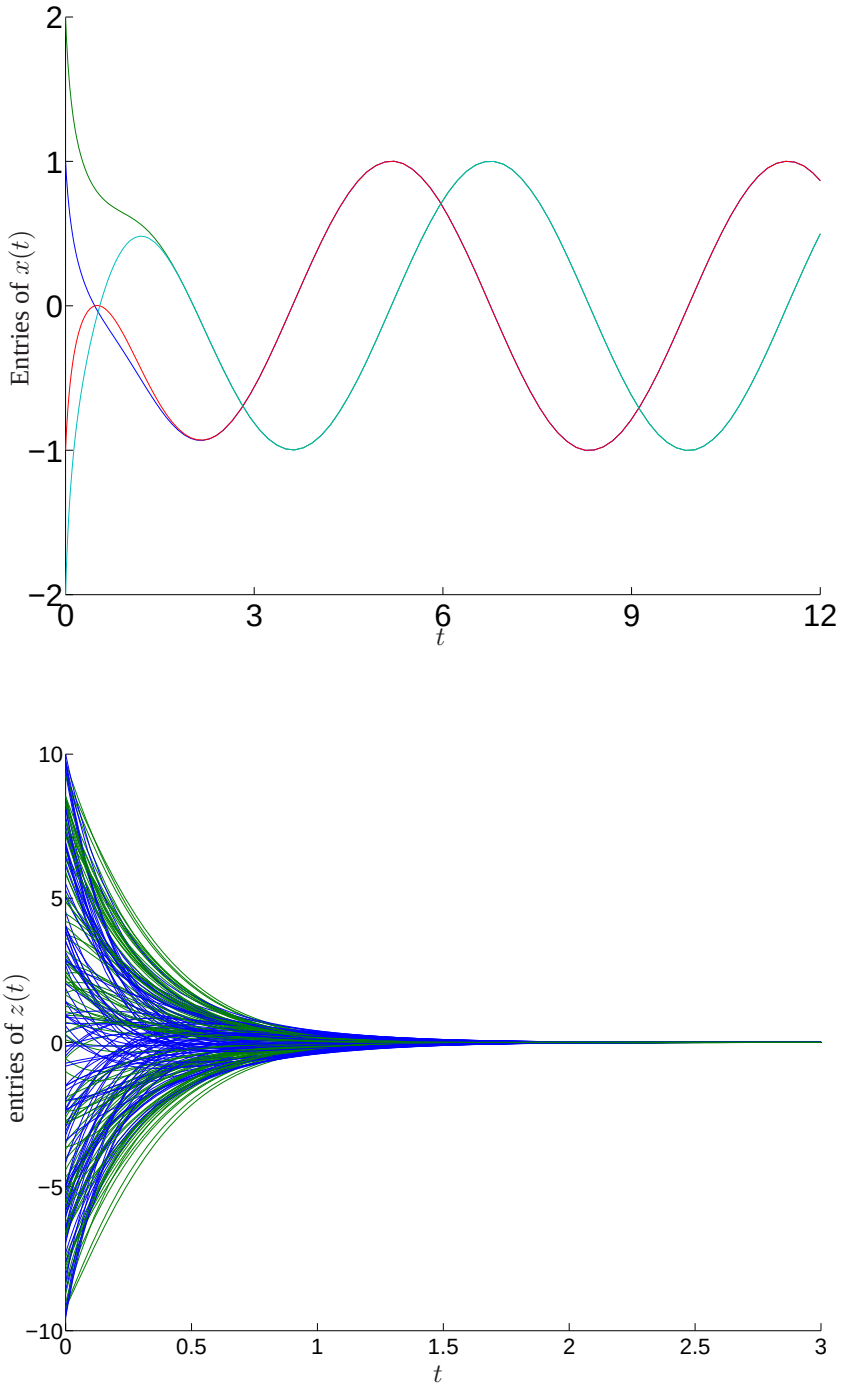


Fig. 3.1: Example for local consensus.

where $x_i = (x_{i1}, x_{i2})^T$, $i = 1, 2$, and

$$c = 1, \ \ \Gamma = I_2, \ \ G = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}, \ \ M = \begin{pmatrix} 0.5 & -0.5 \\ -0.5 & 0.5 \end{pmatrix}.$$

One has that (3.3) holds with $s(t) = (0, 0)^T$. Consider the problem of checking global consensus for this solution. To this end, the LMI condition (3.21) is checked with auxiliary polynomials $u_i(y)$ of degree 2. The result shows that this condition cannot be satisfied by employing quadratic Lyapunov functions. Nevertheless, this condition is feasible with Lyapunov functions of degree 4, in specific the condition holds with $\varepsilon = 0.5$, $u_i(y) = 1 + y_i^T y_i$ and $v(y) = y_1^T y_1 + y_2^T y_2 + (y_1^T y_1)^2 + (y_2^T y_2)^2 - y_{11}^2 y_{21}^2 - y_{12}^2 y_{22}^2$. Therefore, based on Theorem 3.4, global consensus can be achieved.

3.5.3 Example 3

In this case, a coupled jet engines of Moore-Greitzer model is considered for robust local consensus [89]. $f(x)$ in (3.27) shows the intrinsic dynamics of each jet engine as

$$f(x_i) = \begin{pmatrix} -0.5x_{i1}^3 - 1.5x_{i1}^2 - x_{i2} \\ 3x_{i1} - x_{i2} \end{pmatrix}$$

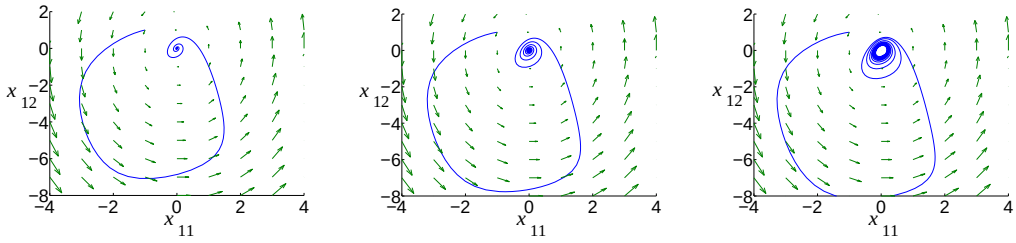
where $x_i = (x_{i1}, x_{i2})^T$, $i = 1, 2$. For this case, a no-stall equilibrium is driven to the origin by following transformation.

$$\begin{cases} x_{i1} &= \tilde{x}_{i1} - 1 \\ x_{i2} &= \tilde{x}_{i2} - x_{\text{co}} - 2. \end{cases}$$

Here, we briefly show the practical meanings of parameters: $\tilde{x}_{i1}$ denotes the mass flow, $\tilde{x}_{i2}$ denotes the pressure rise and x_{co} is a constant. The information exchange between these two jet engines is disturbed by a time-varying uncertainty $\theta(t)$ where

the uncertain weighted adjacency matrix $G(\theta(t))$ is chosen as

$$G(\theta(t)) = \begin{pmatrix} 1 & 2-\theta(t) \\ 1 & 1 \end{pmatrix}.$$



(a) θ=3.2 (b) θ=3.39 (c) θ=3.5

Fig. 3.2: Hopf bifurcation of coupled M-G jet engines.

Around $\theta = 3.392$, a Hopf bifurcation occurs as shown in Fig. 3.2 and robust local consensus can not be achieved when $\theta > 3.392$. Thus let us suppose $\theta \in \Omega = \text{co}\{0, 3.0\}$. Next we will establish whether there is a HPLF such that robust local consensus can be achieved for this chosen range.

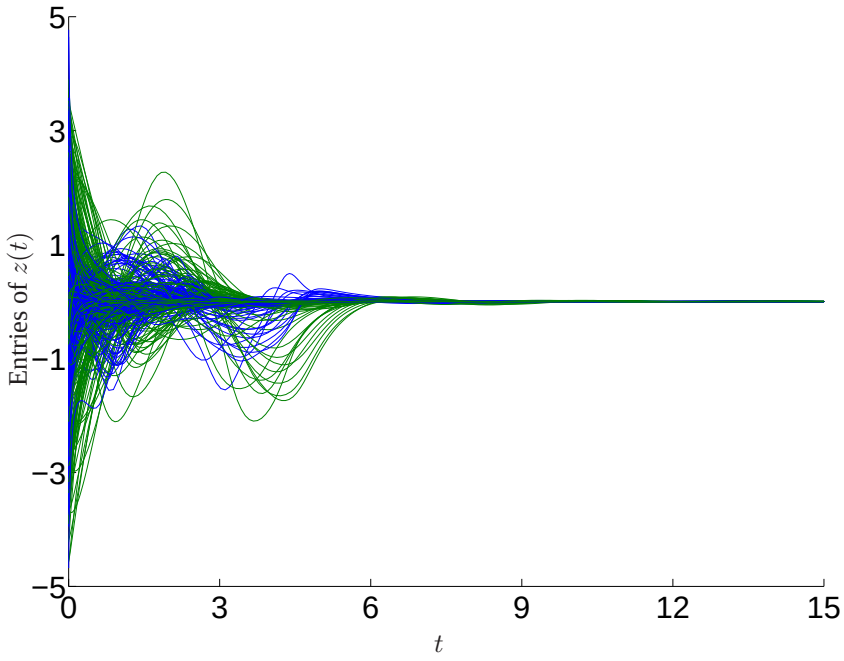


Fig. 3.3: Trajectories of robust local consensus.

Computation results show that we can not find a QLF such that robust local consensus can be achieved where $m = 1$. However, by using a HPLF where $m = 2$, one has that the LMIs (3.38) hold and hence robust local consensus can be achieved from Theorem 3.6. In specific, a HPLF for this case can be constructed that $v(z) = \phi_{\hom}(z, 2)^T I \phi_{\hom}(z, 2)$ with $m = 2$. Figure 3.3 exhibits 100 trajectories of $z(t)$ with the initial conditions $x(0)$ randomly chosen in $[-5, 5]^4$, and $\theta(t)$ randomly chosen in Ω.

3.5.4 Example 4

In this case, we consider (3.24) as an example for robust local consensus with $N = 3$, $n = 1$, $c = 1$, $\Gamma = 1$ and nonlinear function $f(x)$ is given by

$$f(x) = -x - x^3 - x^5.$$

The uncertain weighted adjacency matrix $G(\theta)$ is chosen by

$$G(\theta) = \begin{pmatrix} 1 & 2+\theta & \theta \\ -2-\theta & 1 & 5 \\ \theta & -3 & 1 \end{pmatrix}$$

where $\theta(t) \in \text{co}\{0, 1\}$. One can obtain that (3.26) holds with $s(t) = (0, 0)^T$. By choosing $p_1 = \theta(t)$, it yields that $\widehat{A}(p)$ in (3.33) can be built as

$$\widehat{A}(p) = \begin{pmatrix} -3-2p_1 & 2+p_1 & p_1 \\ -2-p_1 & -4+p_1 & 5 \\ p_1 & -3 & 2-p_1 \end{pmatrix}.$$

Results show that the LMIs (3.38) hold and hence robust local consensus can be achieved according to Theorem 3.6. In this case, the lower bound obtained by (3.41) is tight, i.e., $\hat{\psi}_{2m}^{\infty} = \psi_{2m}^{\infty}$. By applying QLFs, i.e., $m = 1$, one gets $\psi_2^{\infty} = 8.9458$. In comparison, via solving the GEVP (3.44) and exploiting a HPLF, one can find

that robust consensus margin has been significantly expanded, as shown in Table 3.1. By employing bisection method, we obtain that the maximal consensus margin is 13.000 which means by using a HPLF merely with $m = 2$ one can get a very desirable result for this case.

Table 3.1: Consensus margin comparison

	m=1	m=2	m=3	m=4
ψ_{2m}^{∞}	8.9458	12.9397	12.9532	12.9698

3.6 Summary

In this chapter, firstly, we have investigated local and global consensus in MASs with nonlinear dynamics. For local consensus, a method has been provided based on the approximation of the original system into an uncertain polytopic system and on the use of HPLFs, while, for global consensus, another method has been provided based on the pursuit of an appropriate PLF.

In addition, we have considered robust local consensus in MASs with time-varying parametric uncertainties. A novel convex approach has been provided based on the transformation from the original system to an uncertain polytopic system. By using HPLFs, robust local consensus condition can be gained. Moreover, corresponding LMI-based conditions are given by using SMR technique. Polytopic consensus margin has also been estimated by a convex optimization consisting of GEVPs.

Chapter 4

Robust Consensus for Uncertain and Nonlinear Dynamics

4.1 Introduction

Robust consensus with time-varying uncertainty desirably meets the demand of practical implementations and has already been successfully applied in wireless sensor networks and neural networks [90]. In addition, existing uncertain models for consensus protocols usually assume that there exists time-invariant uncertainty or slowly time-varying uncertainty, thus making special academic interests in time-varying uncertainty with bounded variation rate (See Subsection 4.1.2). Furthermore, for adjacency matrix perturbed by uncertain parameters, traditional approaches like eigenvalue analysis are extremely difficult to apply, while it can be suitably tackled with parameter-dependent contraction theory. Last but not the least, even comparing with the prevailing approaches, the QLFs method or parameter-dependent Lyapunov method, the parameter-dependent contraction analysis maintains its advantages in that it does not require an error dynamics (whose construction needs additional assumptions or approximations), and in some circumstances makes the Lyapunov methods as special cases (See Section 4.2).

This chapter considers robust exponential consensus and robust asymptotical

consensus problems affected by time-varying topological uncertainty with bounded variation rate via parameter-dependent contraction analysis. For the first time, to the best of our knowledge, the time-varying topological uncertainty with bounded variation rate is considered in robust consensus problem, making the case with time-invariant uncertainty and the case with time-varying polytopic uncertainty as special ones. An approach of parameter-dependent contraction matrix is proposed by using a general infinitesimal length, which is less conservative than the cases using constant contraction matrix or Lyapunov-like approach. Distinct with nonlinear inequalities provided by traditional methods, this chapter provides tractable conditions of LMIs for robust consensus problem by employing SMR and by parameterizing suitable affine spaces. For robust asymptotical consensus, the lower bound of variation rate margin is estimated via handling GEVPs.

This chapter is organized as follows. Section 4.1 introduces some basic ideas of contraction theory, and robust consensus problems of MASs with rate-bounded polytopic uncertainty are proposed. In Section 4.2, robust consensus conditions are given both for robust exponential consensus and for robust asymptotical consensus. In Section 4.3, HPD-PCM is introduced and robust consensus conditions are proposed in terms of LMIs. Section 4.4 investigates the robust consensus margin. In Section 4.5, some typical examples are given to illustrate our proposed method. Lastly, Section 4.6 summarizes this chapter.

4.1.1 Basics of Contraction Theory

To introduce contraction theory, let us consider a deterministic dynamical system of time-dependent ordinary differential equation as follows

$$\dot{x} = f(x,t),\ x(t_0) = x_0,\ t_0 \geq 0 \tag{4.1}$$

where f is a nonlinear vector field and x is a state vector in a subset of $\mathbb{R}^n$. Suppose that f is continuously differentiable, one can get an exact differential relation

$$\delta\dot{x} = J(x,t)\delta x \tag{4.2}$$

where $J(x,t) = \frac{\partial f}{\partial x}$ stands for the Jacobian of the vector field f, and δx is an infinitesimal change evaluated along a trajectory. δx is also said to be "virtual displacement" which is widely used in classical mechanics and formally deemed as a linear tangent differential form with respect to time [91, 92].

Definition 4.1 *(Contraction) System* (4.1) *is called contracting if there is some* $c > 0$ *such that for every two solutions* $x(t) = \nu(t, 0, \xi)$ *and* $y(t) = \nu(t, 0, \zeta)$ *of System* (4.1)*, initializing from different points, converge exponentially to each other, i.e.* $|x(t) - y(t)| \leq e^{-ct}|\xi - \zeta|$ *where* $f(x,t)$ *is said to be a contracting function.*

Similar with above definition, another one is given here for global asymptotical contraction behavior.

Definition 4.2 *(Asymptotical Contraction) System* (4.1) *is called asymptotically contracting if for every two solutions* $x(t) = \nu(t, 0, \xi)$ *and* $y(t) = \nu(t, 0, \zeta)$ *of System* (4.1)*, initializing from different points, converge asymptotically to each other, i.e.* $\lim_{t\to\infty} |x(t) - y(t)| = 0$ *where* $f(x,t)$ *is said to be an asymptotically contracting function.*

System (4.2) can be considered as a linear time-varying differential equation $\delta\dot{x} = J(t)\delta x$ where $J(t)$ is a fixed function of time. By the Coppel Inequality, one can get an upper bound for the magnitude of its solutions as follows [93],

$$|\delta x|_i \leq |\delta x_0|_i e^{\int_0^t \mu_i(J(\tau))d\tau} \tag{4.3}$$

where $\mu(J)$ is the matrix measure of the Jacobian matrix of f. Following result gives an essential condition about contracting systems which can be tracked down from numerous technical assumptions [91, 94].

Lemma 4.1 *The system* (4.1) *is contracting if there are some matrix measure* $\mu_i(J(x,t))$ *and a positive constant* c *such that*

$$\mu_i(J(x,t)) \leq -c_i \tag{4.4}$$

where the scalar c_i *is called the contraction rate of the system corresponding to vector norm* $|\cdot|_i$.

The matrix measure μ_i corresponding to the induced matrix norms $\|\cdot\|_1$, $\|\cdot\|_2$ and $\|\cdot\|_\infty$ can be obtained in real domain [71, 91] and in complex domain [95]. However, for a particular vector norm and its associated induced matrix norm, it is in general a hard task to get an explicit expression [71]. From next proposition, a clue will be provided on the relationship amongst different matrix measures about contraction. Firstly, let us introduce following Lemma from [96].

Lemma 4.2 *For any two positive real numbers* $p > q > 0$ *and a vector space* $\mathscr{V}$ *with finite dimension* n *with regards to vector norms* $|\cdot|_q$ *and* $|\cdot|_p$*, a relationship can be given by*

$$|x|_p \leq |x|_q \leq n^{(1/q-1/p)}|x|_p. \tag{4.5}$$

Proposition 4.1 *(Equivalence on contraction) For any two positive real numbers* p, q *with* $p > q > 0$, $|\cdot|_q$ *and* $|\cdot|_p$ *are two vector norms on* $\mathscr{V}$*, System 4.1 is contracting for vector norm* $|\cdot|_q$ *with contraction rate* c_q*, which implies that it is also contracting for vector norm* $|\cdot|_p$ *at the same contraction rate with a time-shift* $\psi = \frac{(p-q)\log n}{(pqc_q)}$*, i.e.,*

$$|\delta x|_p \leq |\delta x_0|_p e^{-c_q(t-\psi)}.$$

Proof *From* (4.3)*, one can obtain that*

$$|\delta x|_q \leq |\delta x_0|_q e^{\int_0^t \mu_q(J(\tau))d\tau}.$$

From Definition 4.1, the contraction of System 4.1 for $|\cdot|_q$ yields to

$$|\delta x|_q \leq |\delta x_0|_q e^{-c_q}.$$

Obviously, one can have the contraction rate c_q from the upper bound of matrix measure of system Jacobian as

$$c_q = -\max\{\mu_q(J)\}.$$

According to the equivalence between $|\cdot|_q$ and $|\cdot|_p$ from Lemma 4.2, one has

$$|\delta x|_p \leq n^{(1/q-1/p)}|\delta x_0|_p e^{-c_q t}$$

which can be expressed as

$$|\delta x|_p \leq |\delta x_0|_p e^{-c_q(t-\psi)}$$

where $\psi = \frac{(p-q)\log n}{(pqc_q)}$ denotes a time-shift. □

Considering the equivalence of contraction, we choose Euclidean norm as [91] for ease of description.

4.1.2 Robust Consensus Problems

In this subsection, robust consensus problem with bounded-rate polytopic uncertainties will be introduced. A weighted and directed graph $\mathscr{G} = (\mathscr{A}, \mathscr{E}, G)$ consists of a finite nonempty node set $\mathscr{A} = \{A_1, ..., A_N\}$, a directed edge set $\mathscr{E} \subseteq \mathscr{A} \times \mathscr{A}$, and a weighted adjacency matrix $G \in \mathbb{R}^{N\times N}$. A directed edge from A_j to A_i is denoted as G_{ij} which stands for information can be transmitted from the j-th node to the i-th node but not conversely.

Time-varying parametric uncertainties are considered in this chapter. In spe-

cific, it is supposed that the weighted adjacency matrix G is affected by uncertain parameters $\theta(t) \in \mathbb{R}^a$, denoting the time-varying perturbations from environment to the system dynamics [42, 45]. It satisfies that

$$(\theta(t), \dot{\theta}(t)) \in \Omega = \{(\theta(t), \dot{\theta}(t)) : \theta(t) \in \Lambda_a,\ \dot{\theta}(t) \in \Xi\} \tag{4.6}$$

in which Λ_a is a simplex and Ξ is a polytope given by

$$\begin{cases} \Lambda_a = \{\theta(t) \in \mathbb{R}^a :\ \sum_{i=1}^{a} \theta_i(t) = 1,\ \theta_i(t) \geq 0\} \\ \Xi = \text{co}\{d^{(1)}, ..., d^{(v)}\} \end{cases} \tag{4.7}$$

for some given vectors $d^{(1)}, ..., d^{(v)} \in \mathbb{R}^a$ such that $\sum_{i=1}^{a} d_i^{(j)} = 0,\ \forall j = 1, ..., v$ and $0_a \in \Xi$ where 0_a is a column vector with all a entries being zero.

Remark 4.1 *Since $\dot{\theta}(t)$ belongs to Ξ, one has $\dot{\theta}(t) = \sum_{j=1}^{v} c_j(t) d^{(j)}$ where $c_1(t), ..., c_v(t) \in \mathbb{R}$ satisfy*

$$\begin{cases} \sum_{j=1}^{v} c_j(t) = 1 \\ c_j(t) \geq 0,\ \forall j = 1, ..., v. \end{cases}$$

Thus, one has that

$$\sum_{i=1}^{a} \dot{\theta}_i(t) = \sum_{j=1}^{v} c_j(t) \sum_{i=1}^{a} d_i^{(j)} = 0.$$

Remark 4.2 *The model* (4.6) *has been introduced by [97] and is developed as an extension of models adopted in previous works [98, 99], including various famous models as special cases.*

- *A prevailing model widely adopted by literatures is the uncertainty set of time-invariant polytope. It can be obtained from* (4.6) *by selecting $v = 1$ and $d^{(1)} = 0_a$.*
- *The model of time-varying uncertainty constrained in a polytope can be obtained from* (4.6) *by selecting the scalar a equal to the number of vertices of the polytope and properly choosing the coupling matrices associated to $\theta_1, ..., \theta_a$.*

The uncertain MASs with time-varying uncertainties is under following consensus protocol

$$\dot{x}_i(t) = f(x_i(t)) - b\sum_{j=1}^{N} L_{ij}(\theta(t))\Gamma x_j(t), \;\; i,j = 1,\ldots,N \tag{4.8}$$

where $x_i \in \mathbb{R}^n$ is the state of i-th agent, N is the number of agents, b is the coupling weight, $f(x_i) \in \mathbb{R}^n$ is a nonlinear function, $\Gamma = \mathrm{diag}(\gamma_1,\ldots,\gamma_n) \in \mathbb{R}^{n\times n}$ is a diagonal matrix where $\gamma_i > 0$ means the agents communicating through their i-th states. $L_{ij}(\theta(t))$ is the ij-th entry of the uncertain Laplacian matrix $L(\theta(t)) \in \mathbb{R}^{N\times N}$ given by $L_{ij}(\theta(t)) = -G_{ij}(\theta(t))$ for all $i \neq j$ and by $L_{ii}(\theta(t)) = -\sum_{j=1,\ j\neq i}^{N} L_{ij}(\theta(t))$.

Remark 4.3 *Consensus protocol* (4.8) *is a general form and has a great number of implementations. One special case of time-invariant uncertainty has already successfully applied in voltage analysis of chaotic circuits [100]. As a non-autonomous system with time-varying input, it implies that not only moving equilibrium point is considered, but bounded manifolds like periodic orbit or chaotic oscillator.*

Remark 4.4 *Linear pertubation in communication network is widely employed in literatures [45,64,83]. In this section, we also assume $G_{ij}(\theta(t))$ is a linear function thus the uncertain Laplacian matrix can be displayed as*

$$L(\theta(t)) = L_0 + \sum_{i=1}^{a} \theta_i(t) L_i.$$

Nonlinear coupling with nonlinear disturbs will also be discussed in Section 4.2.

Let us introduce the uncertain MAS (4.1) in compact form as

$$\dot{x}(t) = g(x(t)) - b(L(\theta(t)) \otimes \Gamma)x(t) \tag{4.9}$$

where $x(t) = (x_1(t)^T,\ldots,x_N(t)^T)^T$ and $g(x(t)) = (f(x_1(t))^T,\ldots,f(x_N(t))^T)^T$. Then, the robust consensus problems can be shown as follows.

Problem 4.1 *To establish whether the uncertain dynamical system* (4.9) *achieves*

robust global and exponential consensus, i.e. for any ϵ *there exist positive constants* κ *and* c *such that* $\|x_i(t) - x_j(t)\| \leq \kappa\|x_i(0) - x_j(0)\|e^{-ct}$ *for all* $x_i(0)$, $x_j(0)$, $\theta(t) \in \Omega$ *and* $i, j = 1, \ldots, N$.

Problem 4.2 *To establish whether the uncertain dynamical system* (4.9) *achieves robust global and asymptotical consensus, i.e. for any* ϵ *there exist* $T(\epsilon) > 0$ *such that* $\|x_i(t) - x_j(t)\| \leq \epsilon$ *and* $\lim_{t\to 0} \|x_i(t) - x_j(t)\| = 0$ *for all* $t > T$, $x_i(0)$, $x_j(0)$, $\theta(t) \in \Omega$ *and* $i, j = 1, \ldots, N$.

4.2 Robust Consensus Conditions

In this section, the conception of parameter-dependent contractive matrix will be introduced and corresponding robust consensus conditions will be built via the partial contraction and SMR technique.

An intuitive yet effective way to analyze consensus without topological uncertainty via using contraction theory is the method of partial contraction, where an auxiliary system is adopted and the desired convergence behaviour can be isolated from the overall system dynamics [101].

Lemma 4.3 *Consider a continuously differentiable nonlinear system of the form* $\dot{x} = f(x, x, t)$ *and there is an auxiliary system* $\dot{y} = f(y, x, t)$ *which is contracting with respect to* y. *If a particular solution of the auxiliary* y*-system obtains a smooth specific property, then all trajectories of the original* x*-system have this property exponentially. The original system is said to be partially contracting.*

Let us observe that the virtual system (y-system) has two particular solutions, i.e., $y(t) = x(t)$ sharing this specific property. Provided that all trajectories of virtual system converge exponentially to a specific trajectory, it directly yields that $x(t)$ exponentially verifies these properties.

Example 1 *Let us investigate a consensus problem via using partial contraction.*

Considering a pair of unidirectional coupled oscillators as follows

$$\begin{cases} \dot{x_1} = f(x_1, t) \\ \dot{x_2} = f(x_2, t) + u(x_1) - u(x_2) \end{cases} \tag{4.10}$$

where $x_1,\ x_2 \in \mathbb{R}^n$ *are state vectors,* $f(x_i, t)$ *is the dynamics of uncoupled oscillators and* $u(x_1) - u(x_2)$ *is the coupling force. We can choose a virtual system*

$$\dot{y} = f(y, t) - u(y) + u(x_1).$$

It is clear that $x_1(t) = x_2(t)$ *is a particular solution. On the condition that* $f - u$ *is contracting, consensus can be achieved exponentially.*

Definition 4.3 *Let* $\dot{y} = h(y, \theta, t)$ *be an auxiliary system of* (4.8)*, a symmetric and uniformly positive definite matrix* $M(y, \theta)$ *is called* parameter-dependent contraction matrix *(PD-CM) such that*

$$\mathrm{he}\Big(2\frac{\partial h^T}{\partial y}M + \frac{\partial M}{\partial \theta}\dot{\theta} + \frac{\partial M}{\partial y}\dot{y}\Big) \leq -2\gamma M \tag{4.11}$$

where γ *is a strictly positive scalar. Similarly, a symmetric and uniformly positive definite matrix* $M(y, \theta)$ *is called* parameter-dependent asymptotical contraction matrix *such that*

$$\mathrm{he}\Big(2\frac{\partial h^T}{\partial y}M + \frac{\partial M}{\partial \theta}\dot{\theta} + \frac{\partial M}{\partial y}\dot{y}\Big) \leq -2\gamma I \tag{4.12}$$

where γ *is a strictly positive scalar.*

Lemma 4.4 (From [76]) *Let* $A \in \mathbb{R}^{N\times N}$ *be a symmetric matrix. Product* $W_A^N = (\mathbf{1}_N \cdot \mathbf{1}_N^T) \otimes A$ *is positive semidefinite if and only if* $A \geq 0$*, where* $\mathbf{1}_N$ *is a column vector whose entries are all one.*

Theorem 4.1 *Consider an uncertain system* (4.9)*, an auxiliary system can be built*

as

$$\dot{y}(t) = g(y(t)) - b(L(\theta(t)) \otimes \Gamma)y(t) - W_{P\Gamma}^{N}(\theta)y(t) + W_{P\Gamma}^{N}(\theta)x(t) \qquad (4.13)$$

where $W_{P\Gamma}^{N}(\theta) = (1_N \cdot 1_N^T) \otimes (P(\theta) \cdot \Gamma)$ *and* $P(\theta) \in \mathbb{R}^{n\times n}$ *is a positive definite matrix for all* $(\theta(t), \dot{\theta}(t)) \in \Omega$*. In addition, robust global exponential consensus can be achieved if there is a parameter-dependent contraction matrix* $M(y, \theta)$ *such that* $g(y(t)) - b(L(\theta(t)) \otimes \Gamma)y(t)$ *is contracting.*

Proof

Considering a positive semidefinite matrix $P(\theta)$, for $i, j = 1, \ldots, N$, (4.8) can be equivalently represented as

$$\dot{x}_i(t) = f(x_i(t)) - b\sum_{j=1}^{N} L_{ij}(\theta(t))\Gamma x_j(t) - P(\theta)\sum_{j=1}^{N}\Gamma x_j + P(\theta)\sum_{j=1}^{N}\Gamma x_j. \quad (4.14)$$

Then one can get a compact form from (4.9) such that

$$\dot{x}(t) = g(x(t)) - b(L(\theta(t)) \otimes \Gamma)x(t) - W_{P\Gamma}^{N}(\theta)x(t) + W_{P\Gamma}^{N}(\theta)x(t). \qquad (4.15)$$

Let us consider $W_{P\Gamma}^{N}(\theta)x(t)$ as the system inputs, the auxiliary system (4.13) can be obtained that a particular solution of robust consensus is $y^* = 1_N \otimes y_\infty$ where

$$\dot{y}_\infty(t) = f(y_\infty) - N\ P\Gamma y_\infty + P(\theta)\sum_{j=1}^{N}\Gamma x_j, \ \ i, j = 1, \ldots, N.$$

From Lemma 4.3, the robust consensus of system (4.9) for all $(\theta(t), \dot{\theta}(t)) \in \Omega$ can be achieved and the property $x_1 = ... = x_N$ is able to be verified exponentially if system (4.13) is contracting. Thus, (4.13) is an auxiliary system for system (4.9).

Next, it will show that the auxiliary system (4.13) is contracting if there is a parameter-dependent contraction matrix. A concise proof of exponential convergence of trajectories for contracting system can be found in [102] for an uncertainty-free case. Let y_0 and y_1 be two different points and let $\Upsilon(y, \theta, t)$ be the associated

flow of the auxiliary system (4.13). If there is a parameter-dependent contraction matrix $M(y,\theta)$ given by Definition 4.3, then by the Theorem 2 of [102] one can get

$$D_M(\Upsilon(y_0,\theta,t),\Upsilon(y_1,\theta,t)) \leq e^{(-c/2)t} D_M(y_0,y_1),$$

where D_M is the geodesic distance corresponding to the metric $M(y,\theta)$, and mapping Υ is a strict contraction. Then, from Contraction Mapping Theorem, it implies the flow $\Upsilon(y,\theta,t)$ verifies a specific manifold $y_\infty(t)$ exponentially [70].

Lastly we will show that there is a parameter-dependent contraction matrix M such that (4.13) is contracting. Since $g(y(t)) - b(L(\theta(t)) \otimes \Gamma)y(t)$ is contracting, one can obtain that there exists a matrix $\tilde{M}(\theta)$ such that

$$\begin{aligned} & \mathrm{he}\big(\frac{\partial h^T}{\partial y}\tilde{M} + \frac{\partial \tilde{M}}{\partial \theta}\dot{\theta} + \frac{\partial \tilde{M}}{\partial y}\dot{y}\big) \\ = \;& \mathrm{he}\big(\frac{\partial g^T}{\partial y}\tilde{M}\big) - b\mathrm{he}\big((L(\theta)) \otimes \Gamma)^T\tilde{M}\big) + \frac{1}{2}\mathrm{he}(\frac{\partial \tilde{M}}{\partial \theta}\dot{\theta}) + \frac{1}{2}\mathrm{he}(\frac{\partial \tilde{M}}{\partial y}\dot{y}) \\ \leq \;& -\gamma\tilde{M}. \end{aligned} \tag{4.16}$$

Since Γ is diagonal positive semidefinite and $P(\theta)$ is a positive semidefinite matrix, one can obtain that $\mathrm{he}\big(W^N_{P\Gamma}(\theta)\big) \geq 0$ by Lemma 4.4. Thus, the Riemanian manifold of general infinitesimal length for the auxiliary system (4.13) can be represented by,

$$\begin{aligned} & \frac{\mathrm{d}}{\mathrm{d}t}\delta y^T\tilde{M}(\theta,y)\delta y \\ = \;& \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\delta y^T\mathrm{he}\Big(\tilde{M}(\theta,y)\Big)\delta y \\ = \;& \frac{1}{2}\delta y^T\mathrm{he}\Big(2\frac{\partial h^T}{\partial y}\tilde{M} + \frac{\partial M}{\partial \theta}\dot{\theta} + \frac{\partial \tilde{M}}{\partial y}\dot{y}\Big)\delta y \\ = \;& \frac{1}{2}\delta y^T\mathrm{he}\Big(2\frac{\partial g^T}{\partial y}\tilde{M} - 2b(L(\theta)) \otimes \Gamma)^T\tilde{M} - 2(W^N_{P\Gamma}(\theta))^T\tilde{M} + \frac{\partial M}{\partial \theta}\dot{\theta} + \frac{\partial \tilde{M}}{\partial y}\dot{y}\Big)\delta y \\ = \;& \frac{1}{2}\delta y^T\mathrm{he}\Big(2\frac{\partial g^T}{\partial y}\tilde{M} - 2b(L(\theta)) \otimes \Gamma)^T\tilde{M} + \frac{\partial M}{\partial \theta}\dot{\theta} + \frac{\partial \tilde{M}}{\partial y}\dot{y}\Big)\delta y \\ & -\delta y^T\Big(\big(W^N_{P\Gamma}(\theta)\big)^T\tilde{M}\Big)\delta y - \delta y^T\Big(\tilde{M}W^N_{P\Gamma}(\theta)\Big)\delta y \\ \leq \;& \frac{1}{2}\delta y\mathrm{he}\Big(2\frac{\partial g^T}{\partial y}\tilde{M} - 2b(L(\theta)) \otimes \Gamma)^T\tilde{M} + \frac{\partial M}{\partial \theta}\dot{\theta} + \frac{\partial \tilde{M}}{\partial y}\dot{y}\Big)\delta y \\ \leq \;& -\gamma\delta y^T\tilde{M}(\theta,y)\delta y. \end{aligned}$$

Therefore, the auxiliary system is contracting and hence the proof completes. □

A result can also be gained for robust asymptotical consensus by using parameter-dependent asymptotical contraction matrix as follows.

Theorem 4.2 *Consider an uncertain system* (4.9)*, an auxiliary system can be constructed as* (4.13)*. Furthermore, robust asymptotical consensus can be achieved if there is a parameter-dependent asymptotical contraction matrix* $M(y,\theta)$ *such that* $g(y(t)) - b(L(\theta(t)) \otimes \Gamma)y(t)$ *is asymptotical contracting.*

Proof One can obtain an auxiliary system as (4.13) by similar lines in proof of Theorem 4.1. By Definition 4.3 and let δy_{0-1} be the infinitesimal distance between trajectories starting from y_0 and y_1, one gets

$$\begin{aligned} & \frac{\mathrm{d}}{\mathrm{d}t} D_M(\Upsilon(y_0,\theta,t), \Upsilon(y_1,\theta,t)) \\ = \; & \tfrac{1}{2}\delta y_{0-1}^T \mathrm{he}\Big(\tfrac{\partial h^T}{\partial y} M + M\tfrac{\partial h}{\partial y} + \tfrac{\partial M^T}{\partial \theta}\dot{\theta} + \tfrac{\partial M^T}{\partial y}\dot{y}\Big)\delta y_{0-1} \\ \leq \; & -\gamma \delta y_{0-1}^T \delta y_{0-1}, \;\; \exists \gamma > 0. \end{aligned}$$

where

$$h(y) = g(y(t)) - b(L(\theta(t)) \otimes \Gamma)y(t) - (1_N \cdot 1_N^T) \otimes (P\Gamma)y(t).$$

One can obtain that the trajectories inevitably converge to the set where the Euclidean distance vanishes, i.e., the robust consensus can be achieved for the asymptotical contracting system if there is a parameter-dependent asymptotical matrix.

Provided that $g(y(t)) - b(L(\theta(t)) \otimes \Gamma)y(t)$ is asymptotical contracting, via same lines in proof of Theorem 4.1, the Riemanian manifold of general infinitesimal length can be verified as asymptotically vanishing, i.e.,

$$\frac{\mathrm{d}}{\mathrm{d}t}\delta y^T \hat{M}(\theta, y)\delta y \leq -\gamma \delta y^T \delta y.$$

hence it implies that the auxiliary system is asymptotical contracting. This completes the proof. □

Remark 4.5 *For Theorem 4.1 and Theorem 4.2, note that*

- *The virtual quantity of matrix $P(\theta)$ is to construct the auxiliary system* (4.13)*, satisfying $\left(W_{P\Gamma}^{N}(\theta)^{T}\tilde{M}\right)^{\mathrm{s}} \geq 0$. Note that it has no influence on the actual systems, neither on the specific robust consensus manifold nor on the robust consensus rate. Moreover, matrix $P(\theta)$ in the auxiliary system is not unique.*

- *Comparing with Lyapunov-like method which is broadly adopted in literatures, a system error dynamics has to be constructed as*

$$\dot{\varepsilon}_i = f(\varepsilon_i) + U(\varepsilon_1, ..., \varepsilon_N, \theta),\ i = 1, ..., N$$

where $\varepsilon_i = x_i - \sum_{j=1}^{N} e_j x_j$ with $e_j > 0$ and $\sum_{j=1}^{N} e_j = 1$ and mapping U is a linear function on $\varepsilon_1, ..., \varepsilon_N$ and on θ. However, the nonlinear part $f(\varepsilon_i)$ is difficult to obtain and usually requires a linearization or other approximations (suffered to sorts of assumptions, e.g., global Lipchitz-like condition) which definitely brings conservatism. Furthermore, even though the system error dynamics can be established, a parameter-dependent Lyapunov-like method or Krasovskii's theorem can also be deemed as a special case of contraction theory by selecting

$$V(x, \theta, t) = f(x, t)^{T} M(x, \theta) f(x, t).$$

- *From Lemma 4.1, a more general case can be obtained by using non-Euclidean norms and introducing a general parameter-dependent contraction matrix such that*

$$\begin{cases} M(y, \theta) = M(y, \theta)^{T} \geq 0, \forall \theta \in \Omega. \\ \frac{d}{dt}|M(y, \theta)\delta y|_i \leq -c_i |M(y, \theta)\delta y|_i. \end{cases}$$

Robust consensus conditions can be established by using different vector norms and corresponding induced matrix norms through similar arguments

for Euclidean norm.

- *As a natural extension, nonlinear coupling force is straightly investigated with following consensus protocol.*

$$\dot{x}_i = f(x_i) + \sum_{j=1}^{N} g(x_j - x_i, \theta)$$

where function g is nonlinear on $x_j - x_i$ and linear on θ. Similar results can be gained under additional assumptions that

$$\frac{\partial g(x_j - x_i, \theta)}{\partial (x_j - x_i)} > 0, \ \frac{\partial g(x_j - x_i, \theta)}{\partial \theta} > 0.$$

Nonlinear mapping of θ will be discussed in next subsection.

4.3 Analysis via HPD-PCM

Checking conditions of Theorem 4.1 and Theorem 4.2 are not simple in that they are nonlinear inequality problems with time-varying uncertainties. However, via appropriate parameterizing some affine spaces, SMR technique provides an effective way to solve these problems which amounts to handling with an LMI feasibility test. Indeed, by bringing in a new class of contraction matrix, i.e., HPD-PCM, robust consensus conditions can be provided via solving an LMI feasibility test.

In this chaper, we concern with the robust consensus problems of polynomial nonlinear system. Thus let us introduce the following assumption on $f(x)$.

Assumption 4.1 *The function $f(x_i)$ in* (4.8) *is polynomial.*

Remark 4.6 *One-side global Lipschitz condition (or QUAD condition) is required in an overwhelming number of existing methods for global consensus such as [50]. Nevertheless, the QUAD condition is not fulfilled for simple nonlinearities such as quadratic and cubic functions. By contrast, Assumption 4.1 includes such nonlin-*

earities, and also includes some essential systems like Lorenz-like system, Hamiltonian systems, Guckenheimer system and Rössler system.

Let us define homogeneous parameter-dependent polynomial as follows,

$$m(y,\theta)=\sum_{\substack{q\in\mathbb{N}^n,\ \sum_{i=1}^{n} q_i\leq 2d_y\\ r\in\mathbb{N}^a,\ \sum_{i=1}^{a} r_i=d_\theta}} c_{q,r}y^q\theta^r, \tag{4.17}$$

where $c_{q,r}\in\mathbb{R}$ is the coefficients of monomial $y^q\theta^r$, d_θ of $m(y,\theta)$ denotes the degree in a scalar variables θ, $2d_y$ of $m(y,\theta)$ is the degree in $\tilde{n}$ scalar variables y and $\tilde{n}=Nn$. Thus, a set of homogeneous parameter-dependent polynomial can be given as $\mathscr{H}=\{m(y,\theta):\text{(4.17) holds}\}$. Then, the definition of HPD-PCM can be given as

Definition 4.4 *$M(y,\theta)$ is a* HPD-PCM *if it is a PD-CM and every entry of $M(y,\theta)$ satisfies*

$$M_{ij}(y,\theta)\in\mathscr{H},\ \forall i,j=1,...,\tilde{n}.$$

By a similar way, homogeneous parameter-dependent polynomial asymptotical contraction matrix (HPD-PACM) can be defined by using condition (4.12). Let $R(y,\theta,\dot{\theta},\gamma)$ be a matrix of polynomial as

$$\begin{aligned}&R(y,\theta,\dot{\theta},\gamma)\\ =\ &\Big(\sum_{i=1}^{a}\theta_i\Big)\mathrm{he}\Big(\frac{\partial g^T}{\partial y}M\Big)-b\mathrm{he}\Big((L(\theta)\otimes\Gamma)^TM\Big)+\frac{1}{2}\Big(\sum_{i=1}^{a}\theta_i\Big)^2\mathrm{he}\Big(\frac{\partial M^T}{\partial\theta}\dot{\theta}\Big)\\ &+\frac{1}{2}\Big(\sum_{i=1}^{a}\theta_i\Big)\mathrm{he}\Big(\frac{\partial M^T}{\partial y}g\Big)-\frac{b}{2}\mathrm{he}\Big(\frac{\partial M}{\partial y}(L(\theta)\otimes\Gamma)y\Big)+\gamma\Big(\sum_{i=1}^{a}\theta_i\Big)M.\end{aligned}$$

Thus, condition (4.11) can be displayed in a homogeneous form of degree $d_\theta+1$ in θ since $\sum_{i=1}^{a}\theta_i=1$ for all $\theta\in\Omega$. The condition that $\theta\in\Lambda_a$ can be relaxed to the condition $\theta\in\mathbb{R}_0^a$ by the following lemma.

Lemma 4.5 (From [88]) *The function $H(\theta):\mathbb{R}^a\to\mathbb{R}^{n\times n}$ is a symmetric matrix*

composed of homogenous polynomials with degree d_θ in a scalar variables. Then,

$$H(\theta) > 0 \, \forall \theta \in \Lambda_a \iff H(\mathrm{sq}(\theta)) > 0 \, \forall \theta \in \mathbb{R}_0^{\mathrm{a}}.$$

Lemma 4.6 *Robust exponential consensus of* (4.8) *can be achieved under Assumption 4.1 if there is a positive scalar γ and a HPD-PCM $M(y, \theta)$ such that*

$$\begin{cases} 0 < M(y, \mathrm{sq}(\theta)) \, \forall \mathrm{y} \in \mathbb{R}_0^{\tilde{\mathrm{n}}}, \, \forall \theta \in \mathbb{R}_0^{\mathrm{a}} \\ 0 > R(y, \mathrm{sq}(\theta), \dot{\theta}, \gamma) \, \forall \mathrm{y} \in \mathbb{R}_0^{\tilde{\mathrm{n}}}, \, \forall (\theta, \dot{\theta}) \in \Omega \end{cases} \tag{4.18}$$

Proof *This result can be obtained directly from Definition 4.3, Theorem 4.1 and Lemma 4.5.* □

By the technique of SMR, $M(y, \mathrm{sq}(\theta))$ can be expressed by

$$M(y, \mathrm{sq}(\theta)) = \Psi(\bar{M}, d_y, d_\theta, \tilde{n}) \tag{4.19}$$

where

$$\Psi(\bar{M}, d_y, d_\theta, \tilde{n}) = (*)^T \bar{M} (\phi_{\mathrm{pol}}(y, d_y) \otimes \phi_{\mathrm{hom}}(\theta, d_\theta) \otimes I_{\tilde{n}}),$$

$\phi_{\mathrm{pol}}(y, d_y) \in \mathbb{R}^{l_{\mathrm{pol}}(\tilde{n}, d_y)}$ is a power vector containing all monomials of degree less than or equal to d_y, $\phi_{\mathrm{hom}}(\theta, d_\theta) \in \mathbb{R}^{l_{\mathrm{hom}}(a, d_\theta)}$ is a power vector containing all monomial of degree d_θ, and $l_{\mathrm{pol}}(\tilde{n}, d_y)$, $l_{\mathrm{hom}}(a, d_\theta)$ can be given by (1.6) and (1.11). Symmetric matrix $\bar{M}$ belongs to the set

$$\begin{aligned} \mathscr{M} = \{ & \bar{M}^T = \bar{M} : \Psi(\bar{M}, d_y, d_\theta, \tilde{n}) \text{ only contains} \\ & \text{monomials } \theta^i \text{ with even power } i_k \}. \end{aligned}$$

Lemma 4.7 *The set $\mathscr{M}$ is a linear space of dimension*

$$\begin{aligned}\sigma(\tilde{n}, d_y, d_\theta) &= \tfrac{1}{2}\tilde{n}(l_{\rm pol}(\tilde{n}, d_y)l_{\rm hom}(a, d_\theta)(\tilde{n}l_{\rm pol}(\tilde{n}, d_y)l_{\rm hom}(a, d_\theta) + 1) \\ &\quad -(\tilde{n}+1)(l_{\rm hom}(a, 2d_\theta) - l_{\rm hom}(a, d_\theta))l_{\rm pol}(\tilde{n}, 2d_y))\end{aligned}$$

Proof *Let $\bar{M}_1$ and $\bar{M}_2$ be any matrices in $\mathscr{M}$. It directly follows that for any linear combination of $\bar{M}_1$ and $\bar{M}_2$, one has $c_1\bar{M} + c_2\bar{M}_2 \in \mathscr{M}$, for all $c_1, c_2 \in \mathbb{R}$ such that $c_1 + c_2 = 1$. Thus, one can obtain that $\mathscr{M}$ is an affine space.*

Define

$$a = \tilde{n}l_{\rm pol}(\tilde{n}, d_y)l_{\rm hom}(a, d_\theta),$$

the total number of free entries of $\bar{M} \in \mathbb{R}^{a\times a}$ can be obtained as $\frac{1}{2}a(a+1)$. Let $b \in \mathbb{R}^{\frac{1}{2}a(a+1)}$ be a vector containing the free entries of matrix $\bar{M}$, and define a linear mapping $E : \mathbb{R}^{\frac{1}{2}a(a+1)} \to \mathbb{R}^{a\times a}$ satisfying $E(b) = \bar{M}$. Thus, one has

$$\begin{aligned}&\Psi(\bar{M}, d_y, d_\theta, \tilde{n}) \\ &= (*)^T E(b)(\phi_{\rm pol}(y, d_y) \otimes \phi_{\rm hom}(\theta, d_\theta) \otimes I_{\tilde{n}}) \\ &= (Fb)^T(\phi_{\rm pol}(y, 2d_y) \otimes \phi_{\rm hom}(\theta, 2d_\theta) \otimes I_{\tilde{n}})\end{aligned}$$

where F is a proper transformation matrix. Observe that

$$\mathscr{M} = \{E(b) : b \in \ker(E)\}.$$

It directly follows that

$$\begin{aligned}\dim(\mathscr{M}) &= \dim(\{E(b) : b \in \ker(E)\}) \\ &= \dim(\ker(E)) \\ &= \tfrac{1}{2}a(a+1) - \operatorname{rank}(E).\end{aligned}$$

Let us observe that dimension of $\mathscr{M}$ stems from the entries of some monomials with even power in θ, yielding that

$$\operatorname{rank}(E) = \{\text{number of distinct monomials } c_{i,j}\theta^i y^j \text{ with odd power } i^k\}. \quad (4.20)$$

□

Thus, one can get a complete parametrization of the affine space $\mathscr{M}$ for HPD-PCM of (4.19). Now let us investigate the SMR of $R(y, \theta, \dot{\theta}, \gamma)$. Note that the degree of polynomial $g(y)$ is d_g in y and let

$$d_r = \max(d_g - 1 + 2d_y, 2d_y - 1 + d_g, 2d_y), \tag{4.21}$$

and $2\tilde{d}_r = \text{even}_{+1}(d_r)$ (i.e., $2\tilde{d}_r = d_r$ if d_r is even, and $2\tilde{d}_r = d_r + 1$ if d_r is odd). It follows that,

$$R(y, \theta, \dot{\theta}, \gamma) = \Psi(B(\bar{M}, \dot{\theta}, \gamma) + N, \tilde{d}_r, d_\theta + 1, \tilde{n}) \tag{4.22}$$

where $B(\bar{M}, \dot{\theta}, \gamma)$ is a multilinear function in $\tilde{M}$ and $\dot{\theta}$, i.e., it is linear in $\tilde{M}$ for fixed $\dot{\theta}$ and fixed γ, and is also linear in $\dot{\theta}$ for fixed $\tilde{M}$ and fixed γ, and N is a symmetric matrix belonging to the set

$$\mathscr{N} = \{N^T = N : \Psi(N, \tilde{d}_r, d_\theta + 1, \tilde{n}) = 0\}.$$

Lemma 4.8 *$\mathscr{N}$ is a linear space whose dimension is*

$$\sigma(\tilde{n}, \tilde{d}_r, d_\theta + 1) = \frac{1}{2}\tilde{n}(l(\tilde{n}l + 1) - (\tilde{n} + 1)l_{\text{hom}}(a, 2d_\theta + 2)l_{\text{pol}}(\tilde{n}, 2\tilde{d}_r)) \tag{4.23}$$

where $l = l_{\text{pol}}(\tilde{n}, \tilde{d}_r)l_{\text{hom}}(a, d_\theta + 1)$.

Proof *Similar to the proof of Lemma 4.7 and we omit it here.* □

Interested readers can refer to [84, 88] and its recent developments in robust consensus [81, 82, 87]. The following result provides a sufficient condition which is

a convex problem of LMIs feasibility test.

Theorem 4.3 *The robust exponential consensus of* (4.8) *can be achieved under Assumption 4.1 if there exist matrices* $\bar{M}(\alpha)$, $N(\beta)$ *and a positive scalar* γ *satisfying,*

$$\begin{cases} 0 & < & \bar{M}(\alpha) \\ 0 & > & B(\bar{M}(\alpha), d^j, \gamma) + N(\beta^j),\ \forall j = 1, ..., v. \end{cases} \tag{4.24}$$

where $\bar{M}(\alpha)$ *and* $N(\beta)$ *are linear parametrizations of affine spaces* $\mathscr{M}$ *and* $\mathscr{N}$ *respectively, and* α, β^j *are corresponding free parameters whose dimensions are given by Lemma 4.7 and Lemma 4.8, for all* $j = 1, ..., v$.

Proof *Let us consider the first LMI condition in* (4.24), $\forall \theta \in \mathbb{R}_0^a$, *by pre- and post-multiplying* $(\phi_{\rm pol}(y, d_y) \otimes \phi_{\rm hom}(\theta, d_\theta) \otimes I_{\tilde{n}})^T$ *and* $(\phi_{\rm pol}(y, d_y) \otimes \phi_{\rm hom}(\theta, d_\theta) \otimes I_{\tilde{n}})$, *one can obtain*

$$0 < M(y, {\rm sq}(\theta)).$$

Similarly, from the second condition in (4.24), $\forall y \in \mathbb{R}_0^{\tilde{n}}$ *and* $\forall (\theta, \dot{\theta}) \in \Omega$, *by pre- and post-multiplying* $(\phi_{\rm pol}(y, \tilde{d}_r) \otimes \phi_{\rm hom}(\theta, d_\theta + 1) \otimes I_{\tilde{n}})^T$ *and* $(\phi_{\rm pol}(y, \tilde{d}_r) \otimes \phi_{\rm hom}(\theta, d_\theta + 1) \otimes I_{\tilde{n}})$, *it follows that there is a positive scalar* γ *such that*

$$0 > \Psi(B(\bar{M}, d^j, \gamma) + N(\beta^j), \tilde{d}_r, d_\theta + 1, \tilde{n}),\ \forall j = 1, ..., v.$$

In addition, considering $N(\beta^j) \in \mathscr{N}$, *one has*

$$\Psi(N(\beta^j), \tilde{d}_r, d_\theta + 1, \tilde{n}) = 0,\ \forall j = 1, ..., v.$$

Therefore, it follows that there exists a positive scalar γ *such that*

$$0 > R(y, {\rm sq}(\theta), \dot{\theta}, \gamma),$$

Since Ξ *is a convex hull of vectors* d^j *for* $j = 1, ..., v$, *the condition of Lemma 4.6 holds.* □

An analogous result can be given by using the same approach for robust asymptotical contraction as follows.

Corollary 4.1 *The robust asymptotical consensus of* (4.8) *can be achieved under Assumption 4.1 if it satisfies following condition,*

$$\begin{cases} 0 & < & \bar{M}(\alpha) \\ 0 & > & \widetilde{B}(\bar{M}(\alpha), d^j) + N(\beta^j),\ \forall j = 1, ..., v, \end{cases} \tag{4.25}$$

where

$$\widetilde{R}(y, \theta, \dot{\theta}) = \Psi(\widetilde{B}(\bar{M}, \dot{\theta}) + N, \tilde{d}_r, d_\theta + 1, \tilde{n}), \tag{4.26}$$

and

$$\begin{aligned} & \widetilde{R}(y, \theta, \dot{\theta}) \\ = \ & \Big(\sum_{i=1}^{a} \theta_i\Big) \mathrm{he}\Big(\frac{\partial g^T}{\partial y} M\Big) - \frac{b}{2}\Big((L(\theta) \otimes \Gamma)^T M\Big) + \frac{1}{2}\Big(\sum_{i=1}^{a} \theta_i\Big)^2 \mathrm{he}\Big(\frac{\partial M^T}{\partial \theta}\dot{\theta}\Big) \\ & + \frac{1}{2}\Big(\sum_{i=1}^{a} \theta_i\Big) \mathrm{he}\Big(\frac{\partial M^T}{\partial y} g\Big) - \frac{b}{2} \mathrm{he}\Big(\frac{\partial M}{\partial y}(L(\theta) \otimes \Gamma) y\Big). \end{aligned}$$

4.4 Robust Consensus Performance

Section 4.2 and Section 4.3 propose conditions on which the robust exponential or asymptotical consensus can be established. Follow-up question arises naturally that what is the largest level of polytopic uncertainties where the robustness of asymptotical consensus remains. This section aims to answer this question.

Considering time-varying bounded-rate polytopic uncertainty provided by (4.6), a variation rate margin of robust asymptotical consensus can be introduced for uncertain consensus protocol (4.8). Let η be variation rate margin for system (4.8) as

follows.

$$\begin{aligned}\eta \;\; = \;\; & \sup\Big\{\eta\in\mathbb{R}: \text{(4.8) achieves robust consensus,} \\ & \forall\dot{\theta}\in \mathrm{co}\Big\{\eta d^{(1)},...,\eta d^{(v)}\Big\},\ \forall\theta\in\Lambda_a\Big\}.\end{aligned} \tag{4.27}$$

It is useful to bring in another definition which concerns on the cases that robust asymptotical consensus is guaranteed by a HPD-PACM $M(y,\mathrm{sq}(\theta))$ given by (4.19) for system (4.8) as follows.

Definition 4.5 *Define* $\eta_{\{d_y,d_\theta\}}$ *as* $\{d_y,d_\theta\}$*-HPD-PACM variation rate margin for system* (4.8) *if there exists a HPD-PACM* $M(y,\theta)$ *given by* (4.19) *for system* (4.8) *such that*

$$\eta_{\{d_y,d_\theta\}}=\sup\Big\{\eta\in\mathbb{R}:\dot{\theta}\in\mathrm{co}\Big\{\eta\mathrm{d}^{(1)},...,\eta\mathrm{d}^{(\mathrm{v})}\Big\},\ \forall\theta\in\Lambda_{\mathrm{a}}\Big\}. \tag{4.28}$$

Obviously, $\eta_{\{d_y,d_\theta\}}$ is a lower bound of the variation rate margin, where the robust asymptotical consensus can be ensured by the class of HPD-PACM. In specific, one has

$$\eta_{\{d_y,d_\theta\}}\leq\eta,\ \forall d_y,\ \forall d_\theta.$$

The following results provide a strategy for obtaining a lower bound of $\eta_{\{d_y,d_\theta\}}$ by solving a GEVP problem.

Theorem 4.4 *Define*

$$\hat{\eta}_{\{d_y,d_\theta\}}=\frac{1}{\varsigma^*} \tag{4.29}$$

where ς^* *is the solution of*

$$\varsigma^*=\inf_{\varsigma,\ \bar{M},\ \alpha,\ \beta^{(0)},...,\beta^{(a)}}\varsigma$$

$$\text{s.t.}\begin{cases} 0 \;<\; \varsigma \\ 0 \;<\; \bar{M}(\alpha) \\ 0 \;<\; B_1(\bar{M}(\alpha))+N(\beta^0) \\ 0 \;>\; \varsigma\Big(B_1(\bar{M}(\alpha))+N(\beta^0)\Big) \\ \qquad +B_2(\bar{M}(\alpha),d^i)+N(\beta^i)\ \forall i=1,\ldots,a \end{cases} \tag{4.30}$$

where $\bar{M}(\alpha)$ and $N(\beta)$ are linear parametrization of space $\mathscr{M}$ and $\mathscr{N}$ respectively, $R(y,\theta,\dot{\theta}) = R_1(y,\theta,\dot{\theta}) + R_2(y,\theta,\dot{\theta})$*,*

$$\begin{aligned} & R_1(y,\theta,\dot{\theta}) \\ = & \Big(\sum_{i=1}^{a}\theta_i\Big)\frac{1}{2}\mathrm{he}\Big(\frac{\partial g^T}{\partial y}M\Big)^{\mathrm{s}} - \frac{b}{2}\mathrm{he}\Big((L(\theta)\otimes\Gamma)^T M\Big) \\ & +\Big(\sum_{i=1}^{a}\theta_i\Big)\frac{1}{2}\mathrm{he}\Big(\frac{\partial M^T}{\partial y}g\Big) - \frac{b}{2}\mathrm{he}\Big(\frac{\partial M}{\partial y}(L(\theta)\otimes\Gamma)y\Big), \\ & R_2(y,\theta,\dot{\theta}) \\ = & \Big(\sum_{i=1}^{a}\theta_i\Big)^2\frac{1}{2}\mathrm{he}\Big(\frac{\partial M^T}{\partial\theta}\dot{\theta}\Big), \end{aligned}$$

and

$$\begin{aligned} R_1(y,\theta,\dot{\theta}) &= \Psi\Big(B_1(\bar{M}) + N, \tilde{d}_r, d_\theta + 1, \tilde{n}\Big), \\ R_2(y,\theta,\dot{\theta}) &= \Psi\Big(B_2(\bar{M},\dot{\theta}) + N, \tilde{d}_r, d_\theta + 1, \tilde{n}\Big). \end{aligned}$$

Then $\hat{\eta}_{\{d_y,d_\theta\}}$ is the lower bound of $\eta_{\{d_y,d_\theta\}}$, i.e. $\hat{\eta}_{\{d_y,d_\theta\}} \leq \eta_{\{d_y,d_\theta\}}$.

Proof Suppose that (4.30) holds. Pre- and post-multiplying the second LMI condition in (4.30) by $(\phi_{\mathrm{pol}}(y,d_y)\otimes\phi_{\mathrm{hom}}(\theta,d_\theta)\otimes I_{\tilde{n}})^T$ and $(\phi_{\mathrm{pol}}(y,d_y)\otimes\phi_{\mathrm{hom}}(\theta,d_\theta)\otimes I_{\tilde{n}})$, respectively, one has that

$$0 < \Psi(\bar{M}, d_y, d_\theta, \tilde{n})$$

hence implying $M(y,\theta)$ is positive definite since $(\phi_{\mathrm{pol}}(y,d_y)\otimes\phi_{\mathrm{hom}}(\theta,d_\theta)\otimes I_{\tilde{n}})^T(\phi_{\mathrm{pol}}(y,d_y)\otimes\phi_{\mathrm{hom}}(\theta,d_\theta)\otimes I_{\tilde{n}}) > 0$ for all $y \neq 0$. Then, $R(y,\theta,\dot{\theta})$ for $\dot{\theta} = \varsigma^{-1}\nu^{(i)}$ is given by

$$\begin{aligned} & R(y,\theta,\dot{\theta})|_{\dot{\theta}=\varsigma^{-1}\nu^{(i)}} \\ = & \Psi\Big(B_1(\bar{M}) + \varsigma^{-1}B_2(\bar{M},d^i), \tilde{d}_r, d_\theta+1, \tilde{n}\Big) \\ = & \varsigma^{-1}\Psi\Big(\varsigma B_1(\bar{M}) + B_2(\bar{M},d^i), \tilde{d}_r, d_\theta+1, \tilde{n}\Big). \end{aligned} \tag{4.31}$$

since $N(\beta^i) \in \mathscr{N},\ \forall i = 0,1,...,a$, it follows

$$\begin{aligned} & R(y,\theta,\dot{\theta})|_{\dot{\theta}=\varsigma^{-1}\nu^{(i)}} \\ & = \varsigma^{-1}\Psi\Big(\varsigma(B_1(\bar{M}(\alpha)) + N(\beta^0)) \\ & +B_2(\bar{M}(\alpha),d^i) + N(\beta^i), \tilde{d}_r, d_\theta+1, \tilde{n}\Big). \end{aligned} \tag{4.32}$$

Thus, due to the last condition in (4.30) one has

$$R(y,\theta,\dot{\theta})|_{\dot{\theta}=\varsigma^{-1}\nu^{(i)}} < 0 \ \ \forall i = 1,...,a.$$

Based on this, one can also obtain that there is a HPD-PCM for all $\theta(t)$ in following set

$$\eta_{\{d_y,d_\theta\}} = \sup\Big\{\eta \in \mathbb{R} : \dot{\theta} \in \text{co}\Big\{\varsigma^{-1}d^{(1)},...,\varsigma^{-1}d^{(v)}\Big\},\ \forall\theta \in \Lambda_a\Big\}. \tag{4.33}$$

Therefore, one has $\hat{\eta}_{\{d_y,d_\theta\}} \leq \eta$ which completes this proof. □

4.5 Numerical Examples

4.5.1 Example 1

In this case, a coupled model of Moore-Greitzer jet engines is considered in the no-stall mode [46, 89]. The intrinsic dynamics $f(x)$ in (4.8) is given by

$$f(x_i) = \begin{pmatrix} -0.5x_{i1}^3 - 1.5x_{i1}^2 - x_{i2} \\ 3x_{i1} - x_{i2} \end{pmatrix}$$

where $x_i = (x_{i1}, x_{i2})'$, $i = 1, 2$, x_{i1} relates to the mass flow and x_{i2} associates with the pressure rise. The communications between these two jet engines are affected by a time-varying uncertainty $\theta(t)$. Let us choose the uncertain weighted adjacency matrix $G(\theta(t))$ as

$$G(\theta(t)) = \begin{pmatrix} 1 & 0 \\ 1-2\theta(t) & 1 \end{pmatrix}.$$

For $\theta(t) > 0.721$, the consensus can not be achieved since a Hopf bifurcation takes place as shown in Fig. 4.1 where error states $z(t) = x_1(t) - x_2(t)$. In (a) of Fig. 4.1, $\theta(t) = 0.6,\ \dot{\theta} = 0$, consensus can be achieved where trajectory of agent 1 is

shown in (b). In (c) of Fig. 4.1, $\theta(t) = 0.75,\ \dot{\theta} = 0$, consensus can not be achieved where trajectory of agent 1 is shown in (d). Since $0_a \in \Xi = \text{co}\{d^{(1)}, ..., d^{(v)}\}$, for any η given by (4.27), the robust consensus can not be achieved when $\theta(t) > 0.721$. Hence in this example we consider the parameter bound $0 \leq \theta(t) \leq 0.6$.

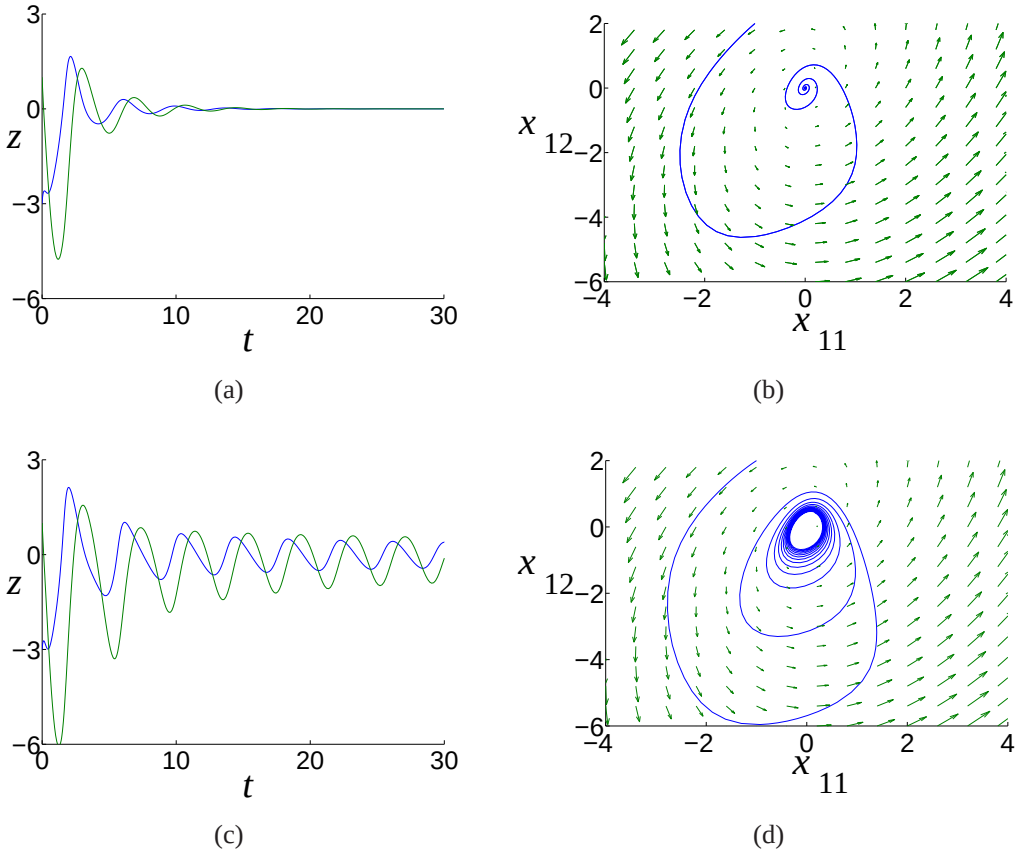


Fig. 4.1: Hopf bifurcation of coupled M-G jet engines.

Let $c = 1$, $\Gamma = I_2$ and a maximum variation rate η of $\theta(t)$ is investigated such that the robust asymptotical consensus can be achieved for any $|\dot{\theta}(t)| \leq \eta$. Hence, Ξ can be expressed as

$$\Xi = \text{co}\left\{ \begin{pmatrix} \frac{\eta}{0.6} \\ -\frac{\eta}{0.6} \end{pmatrix}, \begin{pmatrix} -\frac{\eta}{0.6} \\ \frac{\eta}{0.6} \end{pmatrix} \right\}.$$

Then, we calculate the lower bound $\hat{\eta}$ by using HPD-PCM with $d_\theta = 0, 1, 2, 3$ and $d_y = 1, 2, 3$ as shown in Table 4.1. Comparing with other sufficient conditions proposed by [98] (QLF with affine parameter dependence) and by [46] (Parameter-independent polynomial contraction matrix), the proposed method gen-

Table 4.1: Lower bound $\hat{\eta}$, for some values of d_y and d_θ

d_y/d_θ	0	1	2	3
1	N/A	N/A	N/A	N/A
2	178.3	197.1	207.7	214.1
3	185.8	202.3	211.2	216.4

eralizes these cases and provides a less conservative result by using higher-order HPD-PCM. Specifically, with respect to linear parameter-dependent quadratic Lyapunov function, the robust asymptotical consensus can not be ensured where $d_y = 1$ and $d_\theta = 1$. In addition, the proposed method also has a significant larger bound in contrast with the parameter-independent polynomial contraction matrix where $d_\theta = 0$.

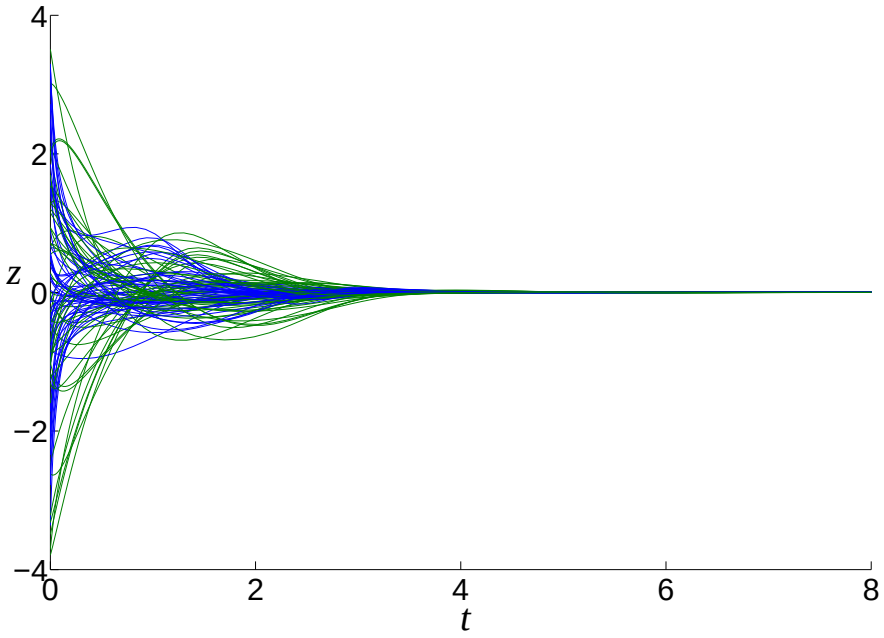


Fig. 4.2: Trajectories of robust consensus.

Figure 4.2 shows that 50 trajectories of $z(t)$ with $\theta(t)$ randomly chosen in Ω and initialized with points randomly chosen in $[-4, 4]^4$.

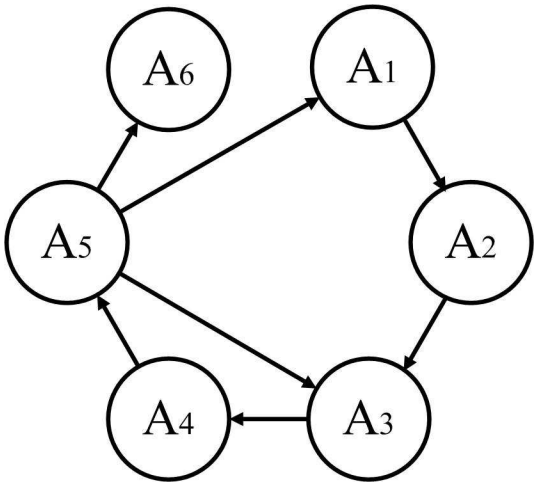


Fig. 4.3: Topology of a six-agent system.

4.5.2 Example 2

In this case, a six-agent system in Figure 4.3 is investigated with intrinsic dynamics in (4.8) as follows

$$f(x_i) = \begin{pmatrix} x_{i2} \\ -3x_{i1} - x_{i2} \end{pmatrix}$$

Let $a = 2$, $n = 2$, $N = 6$, $\Gamma = I_2$ and an uncertain weighted adjacency matrix is given as $G(\theta) = G_0 + \sum_i^a G_i\theta_i$ where

$$G_0 = \begin{pmatrix} 1 & 0 & 0 & 0 & 10 & 0 \\ 5 & 1 & 0 & 0 & 0 & 0 \\ 0 & 8 & 1 & 0 & 20 & 0 \\ 0 & 0 & 6 & 1 & 0 & 0 \\ 0 & 0 & 0 & 8 & 1 & 0 \\ 0 & 0 & 0 & 0 & 9 & 0 \end{pmatrix},$$

$$G_1 = -\begin{pmatrix} 0 & 0 & 0 & 0 & 0.2 & 0 \\ 0.1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$G_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & -0.3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -0.1 & 0 \end{pmatrix}.$$

$\theta = (\theta_1, \theta_2)'$, $\Lambda_2 = \{\theta(t) \in \mathbb{R}^2 : \theta_1 + \theta_2 = 1,\ \theta_1, \theta_2 \geq 0\}$, $v = 2$ and Ξ is chosen to be co$\{d^{(1)}, d^{(2)}\}$ where $d^{(1)} = \eta(1, -1)^T$ and $d^{(2)} = \eta(-1, 1)^T$. Note that this is equivalent to $|\dot{\theta}_i| \leq \eta$ for $i = 1, 2$ and $\dot{\theta}_1 + \dot{\theta}_2 = 0$.

Table 4.2: Comparison of lower bound $\hat{\eta}$ by different approaches with $d_y = 1$.

Approaches/d_θ	0	1	2	3
[98]	N/A	57.34	N/A	N/A
[46]	48.71	N/A	N/A	N/A
This chapter	48.71	59.52	67.13	70.81

Analogous with former example, we calculate the lower bound $\hat{\eta}$ by using HPD-PCM method with $d_y = 1$ and $d_\theta = 0, 1, 2, 3$ as shown in Table 4.2. Comparing with sufficient conditions given by [98] and by [46] , again, the proposed method is verified to be less conservative and has a larger robust asymptotical consensus margin with $d_\theta > 1$. Furthermore, it also displays that by increasing the degree of uncertain parameter d_θ, the conservatism level decreases progressively.

It is worth noting that, in contrast with the approach given by [98], even though same d_θ is considered, proposed method still has a bigger margin $\hat{\eta}$ (also shown

in Table 4.2) in that it completely parameterized corresponding affine spaces while [98] does not.

4.6 Conclusions

In this chapter, robust consensus of MAS with polynomial nonlinear dynamics is considered where the communication network is affected by time-varying polytopic uncertainty with bounded variation rate. Thanks to partial contraction, a novel approach is proposed by adopting a new class of contraction matrix, i.e., HPD-PCM, and conditions for robust exponential consensus and robust asymptotical consensus are both given. Corresponding sufficient conditions have also been provided in terms of LMIs via exploring the parametrizations of related affine sets. Furthermore, we also investigate robust asymptotical consensus margin of the variation rate where the lower bound of this margin can be estimated via solving GEVPs.

In contrast with Parameter Linear-dependent Quadratic Lyapunov Function (PLD-QLF) and parameter-independent polynomial contraction matrix, numerical examples have demonstrated that the proposed method generalize above methods and the conservatism level has apparently decreased by using a higher-order HPD-PCM, in other words, an expanded lower bound of variation rate margin can be found via increasing the values of d_y and d_θ respectively.

Chapter 5

Conclusions and Future Works

This chapter displays the conclusions of this thesis, and provides some possible directions of interests for our future efforts.

5.1 Conclusions

This thesis is concerned with the consensus problems of MASs with different dynamics. More specifically, following kinds of MAS models have been studied and corresponding consensus conditions are given.

1. In chapter 2, robust consensus of MASs with linear dynamics and topological uncertainties is considered, both for continuous-time systems and for discrete-time systems. First, necessary and sufficient conditions are provided for robust first-order consensus and for robust second-order consensus in different cases of positive and non-positive weighted adjacency matrices. In addition, this chapter also investigates robust consensus problem with discrete-time dynamics. Necessary and sufficient conditions are given for robust consensus via finding a polynomial parameter-dependent Lyapunov function. It is also shown that the necessity can be achieved by providing an upper bound on the degree of candidate Lyapunov function required. Then, a necessary and sufficient condition is given for robust first-order consensus with nonnegative weighted adjacency matrices by checking the zeros of a polynomial. Finally,

by expiating SOS technique, these robust consensus conditions can be tested by solving convex optimization problems in terms of LMIs.

2. In chapter 3, HPLF are exploited to solve the consensus problem of MASs. Firstly, local and global consensus in MASs with nonlinear dynamics are investigated. For local consensus, a method has been proposed based on the transformation into an uncertain polytopic system and on the use of HPLFs. Meanwhile, regards to global consensus, another method has been proposed based on the search for a suitable Polynomial Parameter-dependent Lyapunov Function. In addition, we have investigated robust local consensus in MASs with time-varying parametric uncertainties. A novel convex approach has been proposed based on the transformation from the original system to an uncertain polytopic system and on the use of HPLFs. Corresponding LMI-based conditions are obtained by using SMR technique. Polytopic consensus margin has also been investigated by a convex optimization consisting of GEVPs.

3. In chapter 4, robust consensus of multi-agent system with polynomial nonlinear dynamics is considered affected by time-varying polytopic uncertainty with bounded variation rate. Based on partial contraction, a novel approach is proposed by using a new class of contraction matrix, i.e., HPD-PCM, and conditions for robust exponential consensus and robust asymptotical consensus are both provided. Corresponding sufficient conditions have also been proposed in terms of LMIs via exploring the parametrizations of related affine sets. Moreover, we investigate the variation rate for robust asymptotical consensus margin whose lower bound can be estimated via solving GEVPs. Comparing with PLD-QLF and parameter-independent polynomial contraction matrix, numerical examples have shown that the proposed method generalizes above methods and can successfully decrease the conservatism level by using a higher-order HPD-PCM, in other words, an expanded lower bound of variation rate margin can be obtained via increasing the value of d_y and d_θ respectively.

5.2 Future Works

Several possible extensions of research topics considered in this thesis are listed as follows.

1. In our previous works, the consensus design approaches assume that the information of each agent is always available. Nevertheless, in many practical applications, this assumption is not always satisfied in that the states of agents may not be completely accessible due to difficulties in measurements and information transformation. Therefore, it is natural to use the observer-based model where the state information are only available for measured output. In many cases of MASs, implementations have already occurred in problems where dynamic observers are designed to estimate the system states and a feedback controller for each agent can be designed for distributed cooperative control.

2. Based on model of MASs with observer-based controller, another interesting extension is how to design an optimal distributed controller such that the system estimation is as good as possible with considering the environmental noises. Regards to this problem, a number of robust performance criteria can be applied to measure the perturbation against the disturbance inputs, which is usually measured by norms related to the system disturbance input and system responses. A frequently employed measurement is the H_∞ norm method, by which the worst-case effect on the system output can be characterized under the consideration of bounded disturbance.

3. Time-varying topological uncertainty is used to simulate the communication disturbances, thus making the topology of MAS unfixed. Another widely adopted model for chaining topology is MASs with stochastic switching network. It is natural to assume that each agent can only communicate with its neighbours, and as the agent is moving, a time-varying communication networks can be represented by a moving neighbourhood graph, where the edges

of moving neighbourhood graph are set by the location of agents in the lattice. In addition, each agent is assumed to be a random walker and occupies one lattice where information exchange is available for agents in same lattice. The dynamics of moving neighbourhood graph is determined by the motion of agents in lattices, and not affected by the states of agents. This model meets many practical applications in engineering, physics and biology and consensus conditions are expected by using Markov chains, stochastic stability and fast switching theory.

4. As time-delay is a phenomenon commonly encountered in the practical implementations of networked control systems, it is worthy to consider the effects of time-delays in communication for different MAS dynamics. Not merely using the stability theory for time-delay system in existing results of differential equation theory, consensus conditions should be emphasized on the topological condition of MAS in which the consensability is ensured by certain characteristic communication topology. A maximal time-varying time-delay on communication is expected to be estimated by using some convex optimization programming. In addition, other than considering first-order consensus and second-order consensus, a higher-order consensus model can be discussed for a more general case. A detailed analysis for the higher-order consensus algorithms can be an inevitable step to consider more realistic dynamics into the model of individual autonomous agent for future study.

Appendix A

Runtime and Numerical Complexity

Let d, d_θ, a, n and N be the degree of state variable, the degree of θ, the number of uncertain parameters, the number of inner states for a single agent and the total number of agents, respectively. The computational time T_r and number of variables ϖ for some examples in this thesis are shown as Tab. A.1.

Table A.1: Number of variables and runtime for each example

Chapter	Example	Type	Problem	ϖ	T_r (Second)
2	1	d=0, d_θ=2, a=1, N=4, n=1	(2.22)	17	0.4368
	2	d=0, d_θ=2, a=2, N=6, n=1	(2.38)	2402	215.8274
	4	d=0, d_θ=1, a=2, N=6, n=1	(2.95)	247	0.5148
3	1	d=4, d_θ=1, a=1, N=2, n=2	(3.10)	10	0.1680
	2	d=4, d_θ=1, a=1, N=2, n=2	(3.21)	117	1.1260
	3	d=2, d_θ=1, a=1, N=2, n=2	(3.38)	7	0.5304
	4	d=8, d_θ=1, a=1, N=3, n=1	(3.44)	123	2.7839
4	1	d=2, d_θ=2, a=1, N=3, n=1	(4.24)	4653	473.2381
	1	d=4, d_θ=4, a=1, N=3, n=1	(4.30)	29613	3785.9542

Device information are given as follows:

- CPU: Intel Core (TM) i5, 2.67GHz;
- RAM: 4.00 GB;
- Operating System: 64-bit Operating system, Windows 7 Enterprise.

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