Constructing Fuzzy Controllers with B-Spline Models

Jianwei Zhang and Alois Knoll
Faculty of Technology, University of Bielefeld, 33501 Bielefeld, Germany
Email: zhang|knoll@techfak.uni-bielefeld.de

Abstract

We interpret a type of fuzzy controller as an interpolator of B-spline hypersurfaces. B-spline basis functions of different orders are regarded as a class of membership functions (MFs) with some special properties. These properties lead to several interesting conclusions about fuzzy controllers if such membership functions are employed to specify the linguistic terms of the input variables. We show that by appropriately designing the rule base, C^n-continuity of the output can be achieved (n is the order of the B-spline basis functions). The issues of function approximation and heuristic control using such a fuzzy system are also discussed.

1. Introduction

Recently, fuzzy logic control (FLC) has been successfully applied to a wide range of control problems and has demonstrated some advantages, e.g. in efficiency of developing control software, appropriate processing of imprecise sensor data, and real-time characteristics, [9, 12]. However, as pointed out in [3], one obstacle to the wide acceptance for industrial applications is that "it is still not clear how membership functions, defuzzification procedures, ..., contribute, either in combination or as stand-alone factors, to the performance of the FLC". Two important related issues are:

Quality of fuzzy controllers. In practical applications, the smoothness of the controller output is one of the most important design requirements. This applies both to the control of very complex systems such as the speed control of automated trains as well as simple actuators like electrical motors, whose life expectancy directly depends on the smoothness of the controller output. Unfortunately, in general cases, smoothness cannot be guaranteed and is frequently hard to determine for a given controller.

Guidelines for choosing membership functions. Up to now, there exist no convincing guidelines for the successful design of fuzzy controllers. In particular this pertains to the choice of a concrete membership function. In various fuzzy control applications, membership functions of triangular or trapezoidal shape are utilised because of the simplicity of specification and the satisfying results. But the question still remains: can the control performance be improved by choosing a certain set of membership functions?

These two issues can be addressed by comparing B-spline models with a fuzzy logic controller. In our previous work [13], we compared splines and a fuzzy controller with *single-input-single-output* (SISO) structures. In this paper, the *multi-input-single-output* (MISO)[1] controller is considered. Periodical non-uniform B-spline basis functions are interpreted as membership functions. Aspects of function approximation and heuristic control are discussed.

2. Some Previous Work

2.1. Advances in Fuzzy Control

Several authors have shown that fuzzy controllers are universal approximators, [10, 2, 5]. Wang presents a universal approximator by using Gaussian membership functions, product fuzzy conjunction and "centre of average"[2] defuzzification, [10]. Buckley has shown that a modification of Sugeno type fuzzy controllers are universal approximators. Kosko and Dickerson introduced "additive fuzzy systems" to generally describe fuzzy controllers which use the addition of "THEN"-parts of fired rules to determine the crisp output. They then proved that an additive fuzzy system uniformly

[1] A MIMO rule base is normally divided into several MISO rule bases.

[2] Synonyms: Takagi-Sugeno IDM (Inference and Defuzzification Method), Tsukamoto-method, "weighted-mean".

approximates $f : X \to Y$ if X is compact and f is continuous.

Two successful applications in commercial controller and process control are given in [12], one is the OMRON temperature controller (chapt. 3), the other is a gas-fired water heater (chapt. 12). The membership functions are selected as only triangles, and each pair overlaps. Can these be generalised as design rules? The work in [6] shows that the triangular membership functions with 1/2 overlap level produce the zero value of the reconstruction error. Further questions are: Are there other forms of suitable membership functions? Should the overlap of the fuzzy sets for linguistic terms fulfill certain constraints?

2.2. The Popularity of B-Splines

To solve the problem of numerical approximation for smoothing statistical data, "Basis Splines" (B-Splines) were introduced by I. J. Schoenberg [8]. B-splines were used later by R. F. Riesenfeld [7] and W. J. Gordon [4] in CAGD for curve and surface representation. Due to their versatility based on only low-order polynomials and their straightforward computation, B-splines have become more and more popular. Nowadays, B-spline techniques represent one of the most important trends in CAD/CAM areas; they have been extensively applied in modelling free shape curves and surfaces. Recently, splines have also been proposed for neural network modelling and control [1, 11].

Although B-splines have been mainly used in off-line modelling and fuzzy techniques lend themselves to on-line control, some interesting common points can still be found. Our previous paper [13] pointed out that the B-spline basis functions and the membership functions of a linguistic variable are both normalised, overlapping function hulls. Splines and fuzzy controllers possess good interpolation features. The synthesis of a smooth curve with spline functions can easily be associated with the defuzzification process. These points are the main motivation for our work on utilising B-splines to design fuzzy controllers.

3. B-Spline Basis Functions vs. Membership Functions

We consider the membership functions which are used in the context of specifying linguistic terms ("values" or "labels") of input variables of a fuzzy controller. In the following, basis functions of *Non-Uniform B-Splines* (NUBS) are summarised and compared with the membership functions. We also use *B-functions* for the NUBS basis functions.

3.1. NUBS B-Functions

Given a sequence of ordered parameters: $(x_0, x_1, x_2, \ldots, x_m, x_{m+1}, \ldots, x_{m+n})$, the normalised B-functions $N_{i,n}$ of order n are defined as:

$$N_{i,n}(x) = \begin{cases} \begin{cases} 1 & \text{for } x_i \le x < x_{i+1} \\ 0 & \text{otherwise} \end{cases} & \text{if } \; n = 1 \\ \frac{x - x_i}{x_{i+n-1} - x_i} N_{i,n-1}(x) & \\ + \frac{x_{i+n} - x}{x_{i+n} - x_{i+1}} N_{i+1,n-1}(x) & \text{if } \; n > 1 \end{cases}$$

with $i = 0, 1, \ldots, m$.

Three important properties of B-functions are:

partition of unity:	$\sum_{i=0}^{n} N_{i,n}(x) = 1$,
positivity:	$N_{i,n}(x) \ge 0$ for $x \in [0,1]$,
C^{n-2} *continuity:*	If the knots $\{x_i\}$ are pairwise different from each other, then $N_{i,n}(x) \in C^{n-2}$, i.e. $N_{i,n}(x)$ is $(n-2)$-times continuously differentiable.

3.2. Overview of MFs of B-Function Type

The B-functions are employed to specify the linguistic terms, knots are chosen to be different from each other (periodical model). Visually, the selection of n, the order of the B-functions determines the following factors of the fuzzy sets for modelling the linguistic terms, Table 1.

3.3. Partition of the Input Variable into Support Intervals

It is assumed that linguistic terms are to be used to cover $[x_0, x_m]$, the universe of an input variable x of a fuzzy controller. m is chosen according to how fine this input variable should be partitioned by considering an appropriate granularity to achieve a trade-off between the precision of the control/approximation and the complexity of the rule base. If we want to use B-functions $N_{i,n}, i = 1, \ldots, m$ as linguistic terms, then first, $[x_0, x_m]$ is partitioned into m intervals, $[x_i, x_{i+1}]$, $x_i \ne x_{i+1}$, $i = 0, \ldots, m-1$, (Fig. 1). In order to maintain the "partition of unity", some more B-functions should be added at the both ends of $[x_0, x_m]$. They are called *marginal B-functions* and define the *virtual linguistic terms* in the following.

- Marginal B-functions are to be defined on the left end, which need additional $n-1$ intervals adjacent x_0. They are $[x_i, x_{i+1}], i = -n+1, \ldots, -1$. These intervals may be selected as if they have the same

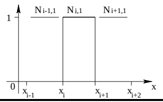
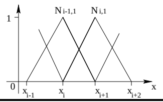
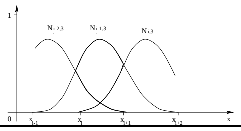
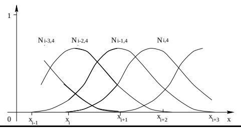
order n	degree	shape	width / overlap
1	0	rectangular 	1
2	1	triangular 	2
3	2	quadratic 	3
4	3	cubic 	4
...	...	...	...

Table 1. The visual effect of fuzzy sets depends mainly on choosing the order of B-functions

span as the symmetrical intervals right to x_0, e.g. $\mid x_{-1}x_0 \mid = \mid x_0x_1 \mid$, etc.

- Marginal B-functions should be also added on the right end, which need still $n-1$ more parameter intervals $[x_i, x_{i+1}], i = m, \ldots, m+n-1$. These intervals can also be selected as having the same span as the symmetrical intervals right to x_m.

In sum, to cover fully the universe $[x_0, x_m]$ with B-functions:

- $m+1$ B-functions of order 2, $N_{-1,2}, \ldots, N_{m-1,2}$, are sequentially labeled as linguistic terms, e.g. $A_1, A_2, \ldots, A_{m+1}$;
- $m+2$ B-functions of order 3, $N_{-2,3}, \ldots, N_{m-1,3}$ which span over $[x_{-2}, x_{m+2}]$ are sequentially labeled as, e.g. $A_0, A_1, \ldots, A_m, A_{m+1}$, where A_0 and A_{m+1} are defined by marginal B-functions $N_{-2,3}$ and $N_{m-1,3}$.
- $m+3$ B-functions of order 4, $N_{-3,4}, \ldots, N_{m-1,4}$, which span over $[x_{-3}, x_{m+3}]$, are sequentially labeled as, e.g. $A_{-1}, A_0, A_1, \ldots, A_m, A_{m+1}$, where $A_{-1}, A_0, A_m, A_{m+1}$ are marginal B-functions.

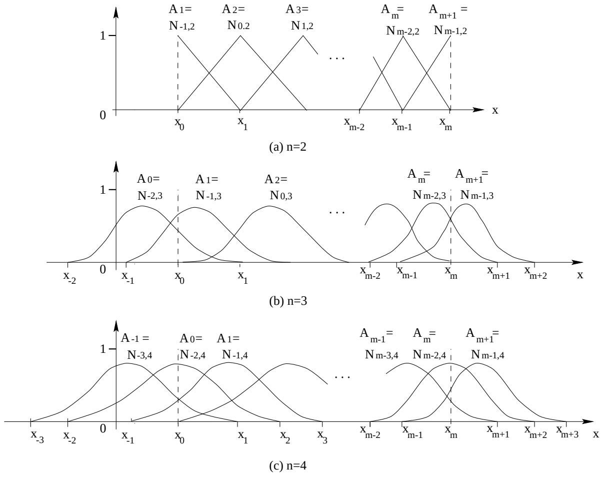


Figure 1. Partition the universe into intervals

- $m+n-1$ B-functions of order n span over $[x_{-n+1}, x_{m+n-1}]$.

4. Construction of the SISO System

Assume that the parameter axis is the input variable x and y is an output variable. x is covered by the B-functions $N_{i,n}$. For specifying linguistic terms of y, the simple fuzzy-singletons y_i are used.

4.1. The Rule Set

We define the *Core Rule Set* (CRS) as *a priori* linguistic rules, which can be described as:

$$CRS = \{Rule(i) : \text{IF } x \text{ is } A_i \text{ THEN } y \text{ is } y_i \mid i = 1, \ldots, s\}$$

If virtual linguistic terms appear in the premise, in order to maintain the output continuity at both ends of the real universe of x, additional rules, called the *Marginal Rule Set* (MRS), are needed to deal with the cases. Marginal rules which contain the left (right) virtual linguistic terms can just use the repeated output value y_1 (y_s).

The whole rule set RS is then: $RS = CRS \cup MRS$.

4.2. Inference and Defuzzification

Given a value x, $x_i < x < x_{i+1}$, it can be seen from the definition of the B-spline basis functions that $N_{i,n}(x), N_{i+1,n}(x), \ldots, N_{i+n-1,n}(x)$ are greater than zero, i.e. exactly n rules fire to non zero degree.

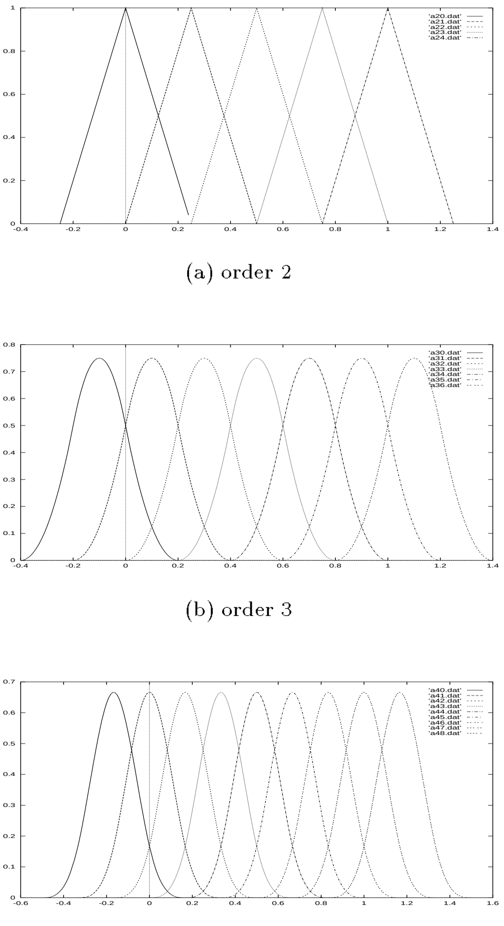
If THEN-parts of fired rules are added and the centre average defuzzifier is used, due to the "partition of unity" property, the crisp value of the output variable is:

$$y = \frac{\sum_{i=-n+1}^{m+n-1} y_i \cdot N_{i,n}(x)}{\sum_{i=-n+1}^{m+n-1} N_{i,n}(x)} = \sum_{i=-n+1}^{m+n-1} y_i N_{i,n}(x) \qquad (1)$$

(1) is exactly the NUBS interpolation of one-dimensional data points. As x varies from x_0 to x_m, y generates a $(n-2)$ times continuously differentiable trajectory.

4.3. An Example

We define a core rule set of 5 rules: $CRS = \{\text{IF } \ x \text{ is } A_i \text{ THEN } y \text{ is } y_i, i = 1, \ldots, 5\}$, where y_1 to y_5 are fuzzy singletons with the following values: 0.5, 1.0, 0.3, 0.55, 0.2.



(a) order 2

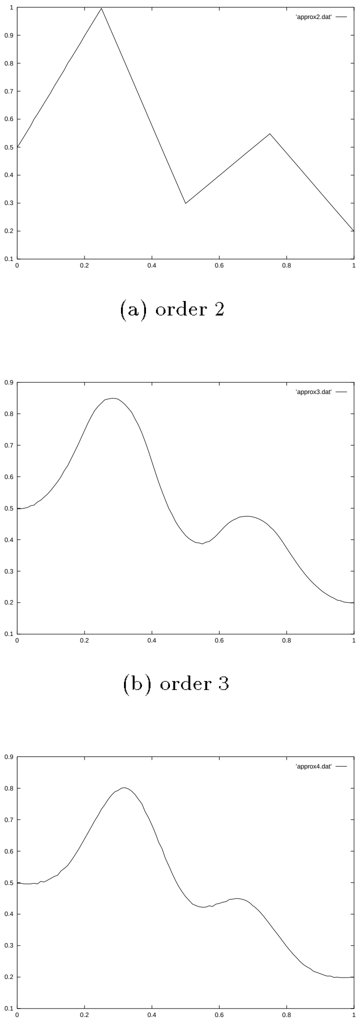
(b) order 3

(c) order 4

Figure 2. Membership functions used for the input variable

Linguistic terms used for input x are shown in Fig. 2 (a)–(c). For B-functions of order 3, one virtual linguistic term is added at the left and one at right end, while for B-functions of order 4, two virtual linguistic term at the left and two at right end. If the two virtual linguistic terms for the case of order 3 are denoted A_0 and A_6, two marginal rules can be constructed by copying the output values y_1 and y_5: $MRS = \{\text{IF } \ x \text{ is } A_0 \text{ THEN } \ y \text{ is } \ y_1; \text{IF } \ x \text{ is } A_6 \text{ THEN } \ y \text{ is } \ y_5\}$.

The curves of the output with respect to input are depicted in Fig. 3 (a)–(c). They are (a): C^0-continuous; (b): C^1-continuous; (c): C^2-continuous.



(a) order 2

(b) order 3

(c) order 4

Figure 3. Output trajectories with respect to the input

5. MISO Fuzzy-Controllers

5.1. Two Inputs and One Output

Consider a fuzzy controller with two inputs x_1, x_2 and one output y. x_1 and x_2 can be independently defined with linguistic terms $\{A_1, \ldots, A_{s_1}\}$ and $\{B_1, \ldots, B_{s_2}\}$. These linguistic terms are modelled with B-functions of order n_1 and n_2 respectively in the procedures illustrated in 3.3. Similarly to 4.1, the *Core Rule Set* can be represented as follows:

$$CRS = \{Rule(i,j) : \text{IF } x_1 \text{ is } A_i(x_1) \text{ and } x_2 \text{ is } B_j(x_2) \text{ THEN } y \text{ is } y_{ij} \mid i = 0, \ldots, s_1, j = 0, \ldots, s_2\}$$

The marginal rule set (MRS) is used for representing all rules which have the virtual linguistic terms in their premises. Their output values are just copied from that of the nearest core rules.

If the "product" conjunction and the "centre average" defuzzifier are used, the output y can be represented as:

$$y = \frac{\sum_{i=-n_1+1}^{s_1+n_1-1} \sum_{j=-n_2+1}^{s_2+n_2-1} y_{ij} \cdot N_{i,n_1}(x_1) \cdot N_{j,n_2}(x_2)}{\sum_{i=-n_1+1}^{s_1+n_1-1} \sum_{j=-n_2+1}^{s_2+n_2-1} N_{i,n_1}(x_1) \cdot N_{j,n_2}(x_2)} \quad (2)$$

$$= \sum_{i=-n_1+1}^{s_1+n_1-1} \sum_{j=-n_2+1}^{s_2+n_2-1} y_{ij} \cdot N_{i,n_1}(x_1) \cdot N_{j,n_2}(x_2) \quad (3)$$

(3) can be associated with the definition of a NUBS surface.

5.2. Examples

Two input variables x_1 and x_2 are covered with three real linguistic terms, represented by $\{A_1, A_2, A_3\}$ and $\{B_1, B_2, B_3\}$, which denote "low", "middle" and "high" respectively. A *core rule set* consisting of 9 rules is shown in Fig. 4(a). On output variable y, fuzzy singletons are defined to represent "VL" (very low), "L" (low), "M" (middle) and "H" (high).

A_1, A_2, A_3 and B_1, B_2, B_3 are defined with adjacent uniform B-functions of order 2, 3 and 4, similar to Fig. 2 (a)-(c). If B-functions of order 3 are used, one virtual linguistic term A_0 (B_0) is added left-adjacent to A_1 (B_1), another A_4 (B_4) is added right-adjacent to A_3 (B_3). Marginal rules which have A_0, A_4, B_0, B_4 in the premise are assigned with the output singletons of the nearest core rule, Fig. 4 (b). For the case of order 4, two virtual linguistic terms for each input variable are added; the construction of the rule base is illustrated in Fig. 4 (c).

The control space on the relation of y with x_1 and x_2 is shown in Fig. 5 (a), (b), (c). The continuity of the three cases is (a): y is continuous; (b): $\partial y/\partial x_1$ and $\partial y/\partial x_2$ are continuous; (f): $\partial^2 y/\partial x_1^2$ and $\partial^2 y/\partial x_2^2$ are continuous.

5.3. The General MISO Case

Generally, rules with q conjunctive terms in the premise are given in the following form:

$\{Rule(i_1, i_2, \ldots, i_q)$: IF ($x_1$ is $N_{i_1,n_1}(x_1)$) and (x_2 is $N_{i_2,n_2}(x_2)$) and ... and (x_q is $N_{i_q,n_q}(x_q)$) THEN y is $y_{i_1 i_2 \ldots i_q}\}$

Under the same conditions in 5.1, the output y of a MISO fuzzy controller is:

$$y = \sum_{i_1=-n_1+1}^{s_1+n_1-1} \cdots \sum_{i_q=-n_q+1}^{m_q+n_q-1} y_{i_1 i_2 \ldots i_q} \prod_{j=1}^{q} \cdot N_{i_j,n_j}(x_j) \quad (4)$$

This is called a *general NUBS hypersurface*. If the B-functions of order $n_1, n_2, \ldots, n_q$ are employed to specify the linguistic terms of the input variables $x_1, x_2, \ldots, x_q$, it can be guaranteed that the output variable y is $(n_j - 2)$ times continuously differentiable with respect to the input variable $x_j, j = 1, \ldots, q$.

We used this type of fuzzy controller in a mobile robot with a modular rule base. Its 6 Infrared sensors and subgoal points are used as input variables, the "speed" and "steer" as output variables. More results are described in [14].

6. Discussions and Conclusions

Some issues related to the construction procedures of a fuzzy controller are:

Definition of membership functions.
The B-functions are piecewise polynomials. If the parameter intervals are equidistant, the membership functions are then uniform B-functions, which can be explicitly represented as polynomial functions. Coefficients of non-uniform B-functions of any order can be computed by a matrix solution. Therefore, they could easily be included in fuzzy development tools to facilitate the modelling of MFs of such type of controller.

Choosing control vertices.
Note that y_i represents also the "control vertices" (de Boor points), which are only identical with the output values for interpolation if the order $n = 2$ (this agrees with the conclusion in [6]). For $n > 2$, control vertices are points near the interpolation point; they "control" the output curve to form a

certain shape inside the convex hull of them. The greater n is, the bigger the difference between control vertices and interpolation points will be.

Normally when rules are formulated using the "IF-THEN" convention, the singleton values of the output are initialised qualitatively in a manner enabling the controller approximately to reach these values; they can be optimised locally by fine-tuning. For this purpose, various adaptive neural-fuzzy methods can be applied, like [10, 5].

Criterion for selecting order n. Obviously, if C^{n-2}-continuity is demanded, the order of B-functions should be at least n. However, a too large value of n leads to more margin linguistic terms and thus more rules, as well as a larger disparity of control vertices and data points. In most applications, C^1– or C^2–continuity is sufficient. Then, B-functions of order 3 and 4 besides these of order 2 with triangular shape could be well suitable for modelling membership functions.

The transformation shown in section 4 and 5 is conceptually important since it provides a quasi construction method for data-approximating using fuzzy controllers. The advantage of the fuzzy control idea over the pure B-spline interpolation lies mainly in its linguistic modelling ability: interpolation data can be prepared using natural language with the help of expert knowledge. Furthermore, the interpolation procedure becomes transparent because it can also be interpreted in fuzzy logic "IF-THEN" form.

Experiments show the feasibility of such type of fuzzy controllers with B-functions as MFs of input variables, singletons as MFs of output variables, "product" as fuzzy conjunction, centre average as defuzzification method. If the rule table is complete, then by adding certain more marginal rules, the smoothness of the controller output can be reached by selecting the proper order of B-functions.

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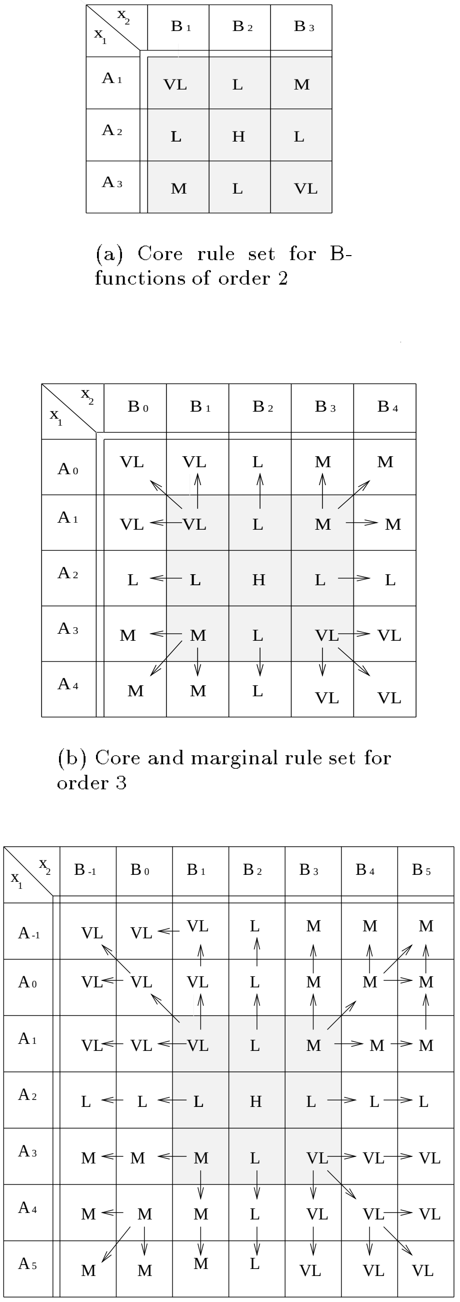


x_1 \ x_2	B_1	B_2	B_3
A_1	VL	L	M
A_2	L	H	L
A_3	M	L	VL

(a) Core rule set for B-functions of order 2

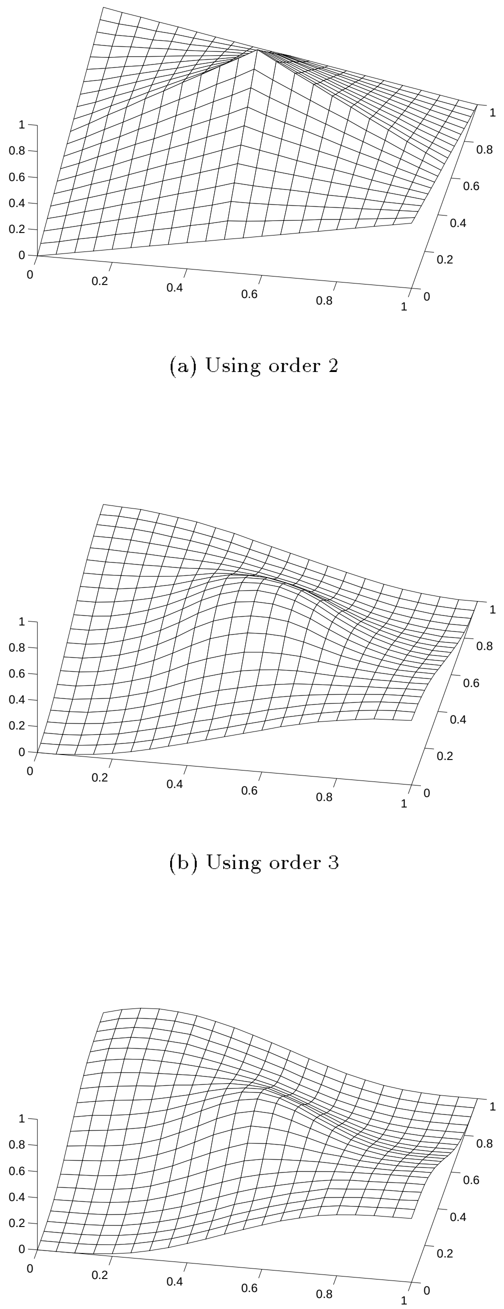
x_1 \ x_2	B_0	B_1	B_2	B_3	B_4
A_0	VL	VL	L	M	M
A_1	VL	VL	L	M	M
A_2	L	L	H	L	L
A_3	M	M	L	VL	VL
A_4	M	M	L	VL	VL

(b) Core and marginal rule set for order 3

x_1 \ x_2	B_{-1}	B_0	B_1	B_2	B_3	B_4	B_5
A_{-1}	VL	VL	VL	L	M	M	M
A_0	VL	VL	VL	L	M	M	M
A_1	VL	VL	VL	L	M	M	M
A_2	L	L	L	H	L	L	L
A_3	M	M	M	L	VL	VL	VL
A_4	M	M	M	L	VL	VL	VL
A_5	M	M	M	L	VL	VL	VL

(c) Core and marginal rule set for order 4

Figure 4. Rule bases for MFs of different orders



(a) Using order 2

(b) Using order 3

(c) Using order 4

Figure 5. The rule bases and the control space of an example with 2 inputs / 1 output